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Abstract. We provide a notion of bargaining set for a finite production economy based on a two-step veto mechanism *à la* Aubin (1979). We show that this bargaining set and the set of Walrasian allocations coincide. At the light of our result we refine Mas-Colell's bargaining set for replicas of a finite economy. Our main result shows the persistence of Anderson *et al.* (1997) non-convergence of the bargaining sets to the set of Walrasian allocations. In addition, we analyze how the restriction on the formation of coalitions affects the bargaining set.

Keywords: Aubin's veto, bargaining sets, coalitions, core.

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1 Introduction

The bargaining set was first introduced by Aumann and Maschler in 1964, on an attempt to inject a sense of credibility and stability to the veto mechanism and hence permitting the implementation of some allocations which otherwise would be formally blocked, although in a non-credible way. In this approach, only objections without counterobjections are considered as credible or justified, and consequently, blocking an allocation becomes more difficult. The bargaining set is defined as the set of feasible allocations that cannot blocked and in a justified way. Therefore the core is contained in the bargaining set.

This original concept of bargaining set was later adapted to atomless economies by Mas-Colell (1989) who, under conditions of generality similar to those required in Aumann's core-Walras equivalence theorem (1964), showed that the bargaining set and the competitive allocations coincide for continuum economies.

In the finite economy framework it is well known that, in general, the core strictly contains the set of Walrasian allocations. Similarly, the translation of Mas-Colell's bargaining set to a finite economy contains the core and then, strictly contains the set of Walrasian allocations.

Debreu and Scarf (1963) formalized the Edgeworth's conjecture (1881), showing that the core shrinks to the set of Walrasian allocations whenever a finite economy is replicated a sufficiently large number of times. However, this core convergence result has been showed not to have the corresponding asymptotic version for the bargaining set. More precisely, Anderson et al. (1997) showed, considering a well-behaved, two-agent economy, that the sequence of bargaining sets of the replicated economies does not converge to the set of Walrasian allocations.

The work by Debreu and Scarf (1963) yields the definition of Edgeworth equilibrium[1] as an attainable allocation whose r-fold repetition belongs to the core of the r-fold replica of the original economy, for any positive integer r. The Edgeworth equilibrium can also be defined as an attainable allocation which cannot be blocked by a coalition in which agents can participate totally or partially with rational rates of participation. The veto system proposed by Aubin allows participation of the agents with any weight in the real unit interval; the corresponding core can be interpreted as a limit notion of Edgeworth equilibrium and Aubin

[1]The concept of Edgeworth equilibrium was defined by Aliprantis et al. (1987). See also Florenzano (1990), where the equivalence between Walrasian allocations and Edgeworth equilibria in general production economies is analyzed.

(1979) showed that it equals the set of Walrasian allocations. Thus, this result for finite economies parallels Aumann's core-Walras equivalence. Considering Aubin's veto in the objection-counterobjection mechanism involved in the definition of the bargaining set, Hervés-Estévez and Moreno-García (2015b, 2017a) obtained a finite version of Mas-Colell's (1989) characterization of competitive allocations. For sake of completeness, this result is set in Section 2.

In Section 3, we define Walrasian objections in a finite production economy framework and show, without using the continuum scenario, that this notion characterizes the justified objections. This result allows us to obtain the Walrasian-bargaining equivalence for finite production economies.

In Section 4, we highlight the differences between the standard objection-counterobjection mechanism, as in Mas-Colell (1989) and Anderson et al. (1997), and Aubin's one. In fact, the counterobjection process *à la* Aubin is not only formally but also economically different from the standard one. Aubin's counter-objection process is specially relevant when agents of the same type must behave coordinately representing the same interests, for instance if individuals of the same type are representatives of an institution, a political party, a trade union or a firm. Indeed, contrary to the standard case, our proposal implies that if one agent participates in an objecting coalition, any other agent of the same type is not allowed to participate in a counterobjection unless she gets a more preferred bundle than the bundle her homologue obtains in the objection.

We define a bargaining set for replicated economies by requiring that members in the counterobjecting coalition improve the bundle obtained by their respective homologue in the coalition that objects. We show that this bargaining set is contained in the one considered in Anderson *et al.* (1997) . Moreover, we highlight that even with this smaller bargaining set, an asymptotic convergence result is not possible.

Finally, to further stress the differences between the Mas-Colell's bargaining set and the one defined in this paper, we analyze how some restrictions on coalitions participating in the objection-counterobjection process affect our bargaining set. In this spirit, Schjødt and Sloth (1994) showed that if one restricts the coalitions to those whose measure is arbitrarily small, then the Mas-Colell's bargaining set becomes strictly larger than the original one. However, we show that the bargaining set we define is not affected by such kind of restriction.

2 Preliminaries: bargaining sets for finite exchange economies

Let $\mathcal{E}$ be an exchange economy with n agents, who trade ℓ commodities. Each consumer i has a preference relation $\succsim_i$ on the set of consumption bundles $\mathbb{R}^\ell_+$, with the properties of continuity, convexity[2] and strict monotonicity. This implies that preferences are represented by utility functions $U_i, i \in N = \{1, \ldots, n\}$. Let $\omega_i \in \mathbb{R}^\ell_{++}$ denote the endowments of consumer i. We can thus summarize the economy as a list $\mathcal{E} = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, i \in N)$.

An allocation x is a consumption bundle $x_i \in \mathbb{R}^\ell_+$ for each agent $i \in N$. The allocation x is feasible in the economy $\mathcal{E}$ if $\sum_{i=1}^n x_i \leqslant \sum_{i=1}^n \omega_i$. A price system is an element of the $(\ell-1)$-dimensional simplex of $\mathbb{R}^\ell_+$. A Walrasian equilibrium for the economy $\mathcal{E}$ is a pair (p, x), where p is a price system and x is a feasible allocation such that, for every agent i, the bundle x_i maximizes the utility function U_i in the budget set $B_i(p) = \{y \in \mathbb{R}^\ell_+ \text{ such that } p \cdot y \leqslant p \cdot \omega_i\}$. We denote by $W(\mathcal{E})$ the set of Walrasian allocations for the economy $\mathcal{E}$.

A coalition is a non-empty set of consumers. An allocation y is said to be attainable or feasible for the coalition S if $\sum_{i \in S} y_i \leq \sum_{i \in S} \omega_i$. The coalition S blocks the allocation x if there exists an allocation y which is attainable for S, such that $y_i \succsim_i x_i$ for every $i \in S$ and $y_j \succ_j x_j$ for some member j in S. The core of the economy $\mathcal{E}$, denoted by $C(\mathcal{E})$, is the set of feasible allocations which are not blocked by any coalition of agents.

It is known that, under the hypotheses above, the economy $\mathcal{E}$ has Walrasian equilibrium and that any Walrasian allocation belongs to the core (in particular, it is efficient).

2.1 A bargaining set for exchange economies

To characterize the Walrasian equilibria in terms of the core, Aubin (1979) enlarges the veto power of coalitions in order to block every non-Walrasian allocation.

An allocation x is blocked in the sense of Aubin by the coalition S via the allocation y if there exist participation rates $\alpha_i \in (0, 1]$, for each $i \in S$, such that

[2]The convexity we require is the following: If a consumption bundle z is strictly preferred to $\hat{z}$, so is the convex combination $\lambda z + (1-\lambda)\hat{z}$ for any $\lambda \in (0, 1)$. This convexity property is weaker than strict convexity and it holds, for instance, when the utility functions are concave.

(i) $\sum_{i \in S} \alpha_i y_i \leqslant \sum_{i \in S} \alpha_i \omega_i$ and (ii) $y_i \succsim_i x_i$, for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$. The Aubin core of the economy $\mathcal{E}$, denoted by $C_A(\mathcal{E})$, is the set of all feasible allocations which cannot be blocked in the sense of Aubin. Under the assumptions previously stated, Aubin (1979) showed that $C_A(\mathcal{E}) = W(\mathcal{E})$.

Definition 2.1 *An Aubin objection to x in the economy $\mathcal{E}$ is a pair (S, y), where S is a coalition that blocks x via y in the sense of Aubin. An Aubin counter-objection to the objection (S, y) is a pair (T, z), where T is a coalition and z is an allocation defined on T, for which there exist $\lambda_i \in (0, 1]$ for each $i \in T$, such that:*

(i) $\sum_{i \in T} \lambda_i z_i \leqslant \sum_{i \in T} \lambda_i \omega_i$

(ii) $z_i \succ_i y_i$ *for every* $i \in T \cap S$ *and* $z_i \succ_i x_i$ *for every* $i \in T \setminus S$.

Definition 2.2 *A feasible allocation belongs to the (Aubin) bargaining set of the finite economy $\mathcal{E}$, denoted by $B(\mathcal{E})$, if it has no justified objection. A justified objection is an objection that has no counterobjection.*

Note that $C_A(\mathcal{E})$, which coincides with the set of Walrasian allocations (Aubin, 1979), is by definition a subset of $B(\mathcal{E})$.

Theorem 2.1 *The bargaining set of the finite economy $\mathcal{E}$ coincides with the set of Walrasian allocations.*

For the proof, we refer to Hervés-Estévez and Moreno-García (2015b, 2017a).

3 Bargaining set in a production economy

To incorporate production into the model, following Debreu and Scarf (1963), we assume that all coalitions have access to the same production possibilities described by a subset Y of the commodity space $\mathbb{R}^\ell$. A point $\xi \in Y$ represents a production plan which can be carried out. Inputs into production appear as negative components of Y and outputs as positive components. The production economy is, thus, denoted by $\mathcal{E}_P = (\mathbb{R}^\ell_+, \succsim_i, \omega_i, Y, \theta_i, i \in N)$, where θ_i represents the consumer i share of participation in the production.

In addition to the assumptions given in the previous section, we will impose in the economy the following conditions:

The production set $Y \subset \mathbb{R}^\ell$ is a convex cone with vertex at the origin, such that $\mathbb{R}^\ell_- \subset Y$ and $Y \cap (-Y) = \{0\}$.

In our production economy, an allocation of commodities is feasible whenever it can be attainable by using the endowments and the production possibilities. That is, an allocation $x = (x_1, \ldots, x_n)$ is feasible if there exists $\xi \in Y$ such that $\sum_{i=1}^n x_i = \xi + \sum_{i=1}^n \omega_i$.

The allocation x is objected by the coalition S if it is possible to find commodity bundles $(y_i, i \in S)$ such that:

(i) $\sum_{i \in S}(y_i - \omega_i) \in Y$

(ii) $y_i \succsim_i x_i$ for all $i \in S$, with strict preference for at least one member of S.

We say that (S, y) is an objection to x.

The core of the economy is defined as the set of all feasible allocations which cannot be blocked or objected by any coalition.

A feasible allocation x is Walrasian if there exists a price system p such that $p \cdot \xi = \max\{p \cdot \xi', \xi' \in Y\}$, where $\xi = \sum_{i=1}^n (x_i - \omega_i)$, and x_i maximizes the preferences of consumer i on her budget set $B_i(p) = \{z \in \mathbb{R}^\ell_+; p \cdot z \leqslant p\omega_i + \theta_i p \cdot \xi\}$.

Observe that under our assumptions, the economy $\mathcal{E}_P$ has Walrasian equilibrium (Debreu, 1959) and that every equilibrium allocation belongs to the core (see Debreu and Scarf, 1963, page 244). Moreover, since Y is assumed to be a cone with vertex at the origin, any equilibrium profit given by $p \cdot \xi$ must be zero, otherwise $p \cdot \lambda\xi > p \cdot \xi$ for all $\lambda > 1$, which is a contradiction. In addition, we remark that the Walrasian equilibrium of the production economy $\mathcal{E}_p$ does not depend on the consumers' share of production profits.

Note also that the particular case where the production set $Y = \mathbb{R}^\ell_-$ is included in our framework. That is, the exchange economy $\mathcal{E}$ is a particular case of the economy $\mathcal{E}_p$.

Next definitions are the respective translation of Mas-Colell's (1989) notions of counterobjection and bargaining set for pure exchange economies with a continuum of agents to the economy with production $\mathcal{E}_P$.

Definition 3.1 *Let (S, y) be an objection to the allocation x. We say that (T, z) is a counterobjection to (S, y) if there exist $z = (z_i, i \in T)$ such that*

(i) $\sum_{i \in T}(z_i - \omega_i) \in Y$

(ii) $z_i \succ_i y_i$ *for all* $i \in S \cap T$ *and* $z_i \succ_i x_i$ *for all* $i \in T \setminus S$.

An objection is justified if it has no counterobjection.

Definition 3.2 *A feasible allocation is in the bargaining set* $B_M(\mathcal{E}_{\mathcal{P}})$ *if it has no justified objection.*

Proposition 3.1 *If* (S, y) *is a justified objection to an allocation* x*, then the allocation* y *is in the core of the restriction of the economy* $\mathcal{E}_{\mathcal{P}}$ *to the coalition* S*.*

Proof. If y is not in the core of the economy restricted to S, there is a coalition of agents $S' \subset S$ and $z = (z_i, i \in S')$ such that (i) $\sum_{i \in S'}(z_i - \omega_i) \in Y$ and (ii) $z_i \succsim_i y_i$ for all $i \in S'$, with strict preference for at least one member of S'. Due to strict monotonicity an continuity of preferences we can change (ii) by (ii)' $z_i \succ_i y_i$ for all $i \in S'$. Thus, (S', z) would be a counterobjection to (S, y).

Next definitions are parallel for $\mathcal{E}_P$ to the Definitions 2.1 and 2.2 that we have stated for exchange economies.

Definition 3.3 *We say that* (S, y) *is an Aubin objection to the allocation* x *if and only if there exist coefficients* $\lambda_i \in (0, 1]$ *such that*

(i) $\sum_{i \in S} \lambda_i (y_i - \omega_i) \in Y$

(ii) $y_i \succsim_i x_i$ *for all* $i \in S$*, with strict preference for at least one member of* S.

The Aubin core of the economy $\mathcal{E}_{\mathcal{P}}$, denoted by $C_A(\mathcal{E}_{\mathcal{P}})$, is the set of all feasible allocations which cannot be objected (blocked) in the sense of Aubin.

Note that it is not necessary to specify the participation rates to the production plan in the feasibility condition.

Definition 3.4 *Let* (S, y) *be an Aubin objection to the allocation* x*. We say that* (T, z) *is an Aubin counterobjection to* (S, y) *if there exist* $z = (z_i, i \in T)$ *and coefficients* $\lambda_i \in (0, 1]$ *such that*

(i) $\sum_{i \in T} \lambda_i (z_i - \omega_i) \in Y$

(ii) $z_i \succ_i y_i$ *for all* $i \in S \cap T$ *and* $z_i \succ_i x_i$ *for all* $i \in T \setminus S$.

Definition 3.5 *A feasible allocation belongs to the (Aubin) bargaining set of the finite economy with production $\mathcal{E}_P$, denoted by $B(\mathcal{E}_P)$, if it has no (Aubin) justified objection. An (Aubin) objection is justified if it has no (Aubin) counterobjection.*

Note that $C_A(\mathcal{E}_\mathcal{P})$ is, by definition, a subset of $B(\mathcal{E}_P)$.

In which follows we provide a characterization of (Aubin) justified objections in terms of prices.

Definition 3.6 *Let x be an allocation in the economy $\mathcal{E}_P$. An (Aubin) objection (S, y) to x is said to be Walrasian if there exists a price system p such that (i) $p \cdot \xi = \max\{p \cdot \xi'; \xi' \in Y\} = 0$, where $\xi = \sum_{i \in S} \lambda_i(y_i - \omega_i)$, (ii) $p \cdot v \geqslant p \cdot \omega_i$ if $v \succsim_i y_i$, $i \in S$ and (iii) $p \cdot v \geqslant p \cdot \omega_i$ if $v \succsim_i x_i$, $i \notin S$.*

We remark that, under the assumptions of monotonicity of preferences and strict positivity of the endowments, we know that any price system p that sustains a Walrasian objection is such that $p \gg 0$, and therefore conditions *(ii)* and *(iii)* above can be written as follows: $v \succ_i y_i$ implies $p \cdot v > p \cdot \omega_i$, for $i \in S$ and $v \succ_i x_i$ implies $p \cdot v > p \cdot \omega_i$ for $i \notin S$.

Theorem 3.1 *Let x be a feasible allocation in the economy $\mathcal{E}_P$. Then, an Aubin objection to x is justified if and only if it is a Walrasian objection.*

Proof. Let (S, y) be a Walrasian objection *à la Aubin* to x. Assume that (T, z) is a counterobjection in the sense of Aubin to (S, y). Then, there exist coefficients $\lambda_j \in (0, 1]$ for each $j \in T$, such that: $\sum_{j \in T} \lambda_j(z_j - \omega_j) = \xi \in Y$; $z_j \succ_j y_j$ for every $j \in T \cap S$ and $z_j \succ_j x_j$ for every $i \in T \setminus S$. Since (S, y) is a Walrasian objection at prices p, we have that $p \cdot z_j > p \cdot \omega_j$, for every $j \in T \cap S$ and $p \cdot z_j > p \cdot \omega_j$, for every $j \in T \setminus S$. This implies $p \cdot \sum_{j \in T} \lambda_j z_j > p \cdot \sum_{j \in T} \lambda_j \omega_j$, which is in contradiction with $p \cdot \xi \leq 0$. Thus, we conclude that (S, y) is a justified objection.

To show the converse, let (S, y) be a justified objection to x and let $a = (a_1, \ldots, a_n)$ be an allocation (not necessarily feasible) such that $a_i = y_i$ if $i \in S$ and $a_i = x_i$ if $i \notin S$. For every consumer i define $\Gamma_i = \{z \in \mathbb{R}^\ell | z + \omega_i \succ_i a_i\}$ and let Γ be the convex hull of the union of the sets $\Gamma_i, i \in N$.

Let us show that $\Gamma \cap Y$ is empty. Assume that $\xi \in \Gamma \cap Y$. Then, there is $\lambda = (\lambda_i, i \in N) \in [0, 1]^n$, with $\sum_{i=1}^n \lambda_i = 1$, such that $\xi = \sum_{i=1}^n \lambda_i z_i$, with $z_i \in \Gamma_i$. This implies that the coalition $T = \{j \in N \mid \lambda_j > 0\}$ counterobjects (S, y) via the allocation $\hat{z}$, where $\hat{z}_i = z_i + \omega_i$ for each $i \in T$. Indeed, $\sum_{j \in T} \lambda_j(\hat{z}_j - \omega_j) = \xi \in Y$. Moreover, since $z_i \in \Gamma_i$ for every $i \in T$, $\hat{z}_i \succ_i y_i$ for every $i \in T \cap S$ and $\hat{z}_i \succ_i x_i$ for every $i \in T \setminus S$. This is a contradiction.

Thus, $\Gamma \cap Y = \emptyset$. Therefore, there exists a hyperplane that separates the convex sets Γ and Y. That is, there exists a price system p such that $p \cdot z \geqslant 0$ for every $z \in \Gamma$ and $p \cdot \xi \leqslant 0$ for all $\xi \in Y$. For $i \in S$, note that if $v \succ_i y_i = a_i$, i.e., $v - \omega_i \in \Gamma_i \subset \Gamma$, then $p \cdot v \geqslant p \cdot \omega_i$. Note also that, by continuity and monotonicity of preferences, $v \succsim_i y_i$ also implies $p \cdot v \geqslant p \cdot \omega_i$. For $i \notin S$, a similar argument shows that if $v \succsim_i x_i$, then $p \cdot v \geqslant p \cdot \omega_i$. Moreover, the fact that $p \cdot y_i \geqslant p \cdot \omega_i$, for every $i \in S$, implies $p \cdot \sum_{i \in S} \lambda_i(y_i - \omega_i) \geqslant 0$. Since y is attainable in the sense of Aubin for S, we also have $\sum_{i \in S} \lambda_i(y_i - \omega_i) = \bar{\xi} \in Y$. Then, $p \cdot \bar{\xi} = 0$. Therefore, we conclude that (S, y) is a Walrasian objection.

Q.E.D.

Remark. The approach of the proof above constitutes a characterization of Aubin justified objections in terms of prices for finite economies. We point out that an Aubin objection is in fact an equal treatment objection to an equal treatment allocation in a continuum economy with a finite number of types of agents.

Given the finite production economy $\mathcal{E}_P$, let $\mathcal{E}_P^c$ be the continuum economy where the set of agents is $I = [0,1] = \bigcup_{i=1}^n I_i$, endowed with the Lebesgue measure μ, being $I_i = \left[\frac{i-1}{n}, \frac{i}{n}\right)$ if $i \neq n$ and $I_n = \left[\frac{n-1}{n}, 1\right]$. Every $t \in I_i$ has endowments $\omega(t) = \omega_i$ and preferences $\succsim_t = \succsim_i$, that is, all the consumers in I_i are of the same type i. As in the finite case, every coalition has access to any production plan $\xi \in Y$. The share in the production profits of each agent $t \in I$, is given by $\theta(t)$, in such a way that $\int_I \theta(t) d\mu(t) = 1$.

An allocation is any integrable function f defined on I with values in $\mathbb{R}_+^\ell$. An allocation f is feasible in $\mathcal{E}_P^c$ if $\int_I (f - \omega) \in Y$. A feasible allocation is competitive if there exists a price system $p > 0$ such that $0 = p \cdot \int_I (f - \omega) = \max\{p \cdot y', y' \in Y\}$ and $f(t) \in B_t(p) = \{z \in \mathbb{R}_+^\ell, p \cdot z \leqslant p \cdot \omega(t)\}$ is such that $f(t) \succsim_t z$ for every $z \in B_t(p)$ and for almost all $t \in I$.

Observe that $x = (x_1, \ldots, x_n)$ is a Walrasian allocation in the finite economy $\mathcal{E}_P$ if and only if f_x is a competitive allocation in the continuum economy $\mathcal{E}_P^c$, where f_x is the step function $f_x(t) = x_i$ for every $t \in I_i$. Moreover, if f is a competitive allocation in $\mathcal{E}_P^c$, the allocation $x^f = (x_1^f, \ldots, x_n^f)$ given by $x_i^f = \frac{1}{\mu(I_i)} \int_{I_i} f(t) d\mu(t)$ is a Walrasian allocation in the finite economy $\mathcal{E}_P$. (See García-Cutrín and Hervés-Beloso, 1993, for exchange economies).

Mas-Colell (1989) stated a notion of bargaining set for continuum exchange economies and showed that it characterizes the competitive allocations. The translation of Mas-Colell's definition of bargaining set for the production economy $\mathcal{E}_P^c$ is as follows:

An objection to the allocation f in the economy $\mathcal{E}_P^c$ is defined by (S, g), where S is a coalition of agents (a positive measure subset of I) and g is an attainable allocation for S (i.e., $\int_S (g(t) - \omega(t))d\mu(t) \in Y$) such that $g(t) \succeq_t f(t)$ for every $t \in S$ and $\mu(\{t \in S; g(t) \succ_t f(t)\}) > 0$.

A counterobjection to the objection (S, g) is defined by (T, h), where h is a feasible allocation for the coalition T and such that $h(t) \succ_t g(t)$ for every $t \in T \cap S$ and $h(t) \succ_t f(t)$ for every $t \in T \setminus S$.

Following Mas-Colell (1989), the objection (S, g) to the allocation f is a Walrasian objection if there is a price system p such that $p \cdot \bar{\xi} = \max\{p \cdot \xi; \xi \in Y\}$, where $\bar{\xi} = \int_S (g(t) - \omega(t))d\mu(t) \in Y$, and the following properties hold for almost all $t \in I$: (i) if $t \in S$, then $p \cdot v \geqslant p \cdot \omega(t)$ for every $v \in \mathbb{R}_+^\ell$ with $v \succsim_t g(t)$ and (ii) if $t \in I \setminus S$, then $p \cdot v \geqslant p \cdot \omega(t)$ for every $v \in \mathbb{R}_+^\ell$ with $v \succsim_t f(t)$.

Next lemma is the analogue for the production economy $\mathcal{E}_P^c$ of Proposition 2 in Mas-Colell (1989), stated for a pure exchange economy setting.

Lemma 3.1 *Every noncompetitive allocation in $\mathcal{E}_P^c$ has a Walrasian objection.*

For the proof it suffices to adapt the proof of the aforementioned Proposition 2 in Mas-Colell (1989) to the production economy we consider. (See also Liu and Zhang, 2016).

Next theorem, which is a consequence of our Theorem 3.1, is the main result in this Section.

Theorem 3.2 *The bargaining set of the finite production economy $\mathcal{E}_P$ coincides with the set of Walrasian allocations.*

Proof. Every Walrasian allocation in $\mathcal{E}_P$ belongs to the core (see Debreu and Scarf, 1963, page 244) and then it is in the bargaining set $B(\mathcal{E}_P)$.

In order to show the converse, suppose that x is not a Walrasian allocation. Then, the associated allocation f_x in the atomless economy $\mathcal{E}_P^c$ is not competitive and thus, by Lemma 3.1, there is a Walrasian objection $(\hat{S}, g)$ to f_x. This means that there is a price system p such that $p \cdot \bar{\xi} = 0 = \max\{p \cdot \xi; \xi \in Y\}$, where

$\bar{\xi} = \int_{\hat{S}}(g(t) - \omega(t))d\mu(t) \in Y$, and for almost all $t \in I$: (i) $p \cdot v \geqslant p \cdot \omega(t)$ for $v \in \mathbb{R}^\ell_+$ if $v \succsim_t g(t)$ and $t \in \hat{S}$; (ii) $p \cdot v \geqslant p\omega(t)$ for $v \in \mathbb{R}^\ell_+$ if $v \succsim_t f_x(t)$ and $t \in I \setminus \hat{S}$.

Let $S = \{j \in \{1, \ldots, n\}; \mu(\hat{S} \cap I_j) \neq \emptyset\}$ and for any member $j \in S$, let $z_j = \frac{1}{\mu(\hat{S}\cap I_j)} \int_{\hat{S}\cap I_j} g(t)d\mu(t)$. Then, we have that $\sum_{j\in S} \mu(\hat{S}\cap I_j)(z_j - \omega_j) = \int_{\hat{S}}(g(t) - \omega(t))d\mu(t) = \bar{\xi} \in Y$ and thus z is an Aubin feasible allocation for the coalition S. Next, we will see that the price p guarantees that z is a Walrasian objection to x in $\mathcal{E}_P$. In fact, if $v \succsim_j z_i, i \in S$, then $v \succsim_t g(t)$ for every t in a positive measure subset of $\hat{S} \cap I_i$ (see Lemma in García-Cutrín and Hervés-Beloso, 1993, page 580). Since $(\hat{S}, g)$ is a Walrasian objection we have that $p \cdot v \geqslant p \cdot \omega_j$. Moreover, if $i \in N \setminus S$ then, if $v \succsim_i x_i = f_x(t), t \in I_i$ we have that $p \cdot v \geqslant p \cdot \omega_i$. Since $\bar{\xi}$ maximizes profits at price p, we conclude that x has a Walrasian objection and thus, it is out of the bargaining set $B(\mathcal{E}_P)$.

Q.E.D.

4 Some remarks on bargaining sets

4.1 Objection mechanism in replicated economies

In a finite economy framework it is well known that, in general, the core strictly contains the set of Walrasian allocations. In order to provide foundations to Walrasian mechanism, Debreu and Scarf (1963) formalized Edgeworth's conjecture showing that the core and the set of Walrasian allocations become arbitrarily close whenever a finite economy is replicated a sufficiently large number of times. This result yields the definition of Edgeworth equilibrium for an economy with a finite number of agents ($\mathcal{E}$ or $\mathcal{E}_P$) as a feasible consumption plan whose r-fold repetition belongs to the core of the r-fold replica of the original economy, for any positive integer r.

Following Debreu-Scarf, for each positive integer r, the r-fold replica economy $r\mathcal{E}$ of $\mathcal{E}$ is a new economy with rn agents indexed by ij, $j = 1, \ldots, r$, such that each consumer ij has a preference relation $\succsim_{ij} = \succsim_i$ and endowments $\omega_{ij} = \omega_i$. That is, $r\mathcal{E}$ is an economy with r agents of type i for every $i \in N$. Given a feasible allocation x in $\mathcal{E}$, the replica allocation, rx, is the corresponding equal treatment allocation in $r\mathcal{E}$, which is given by $(rx)_{ij} = x_i$ for every $j \in \{1, \ldots, r\}$ and $i \in N$.

A feasible allocation x is not Walrasian in the economy $\mathcal{E}$ if and only if there are a coalition of consumers S, a consumption plan y_i for $i \in S$ and an integer number r_i of consumers identical to consumer i, such that the coalition formed by r_i agents of type i blocks x in the r-replica ($r \geqslant \max r_i$) of the economy $\mathcal{E}$. That is, (i) $\sum_{i\in S} r_i y_i \leqslant \sum_{i\in S} r_i \omega_i$ and (ii) $y_i \succsim_i x_i$, for every $i \in S$ and $y_j \succ_j x_j$ for some $j \in S$ (Debreu and Scarf, 1963, page 245, Theorem 3).

Observe that dividing the inequality in (i) by $\max\{r_i\}$, we deduce that a Walrasian allocation can also be defined as an attainable allocation that cannot be blocked (objected) by an Aubin coalition with rational rates of participation. On the other hand, when the participation rates are rational numbers, the veto mechanism in the sense of Aubin is the standard veto system in the sequence of replicated economies of the original economy $\mathcal{E}$. To be precise, if the parameters defining the participations rates of each member in a blocking coalition S are rational numbers $\alpha_i = \frac{p_i}{q_i}$, then there are integers $r_i, i \in S$ and $r \geqslant \max\{r_i, i \in S\}$, such that $\alpha_i = r_i/r$ for every $i \in S$. That is, we can say that the coalition formed by r_i agents of type i blocks the allocation rx in the replicated economy $r\mathcal{E}$.

4.2 Mas-Colell's bargaining set in replicated economies and Aubin's bargaining set

In spite of the analogies between Aubin's veto and the the standard veto on the sequence of replicated economies of a given finite (exchange or production) economy, the difference between the corresponding bargaining sets is relevant. The aim of this section is to analyze and highlight such differences. To this end, it is enough to consider an exchange economy since production economies contain exchange ones as a particular case.

- A standard objection $(\bar{S}, y)$ to the allocation rx in the replicated economy $r\mathcal{E}$ is defined by the coalition $\bar{S}$, formed by $1 \leqslant r_i \leqslant r$ consumers of type $i \in S \subset N$ and consumption plans y_{ij} with $i \in S$ and $j \in \{1, \ldots, r_i\}$, such that (i) $\sum_{ij\in\bar{S}} y_{ij} \leqslant \sum_{i\in S} r_i \omega_i$ and (ii) $y_{ij} \succsim_i x_i$, for every $ij \in \bar{S}$ and $y_{ij} \succ_i x_i$ for some $ij \in \bar{S}$. Under the convexity assumption on preferences, in condition (ii) above, we can assume without lost of generality that $y_{ij} = y_i$ for all j and thus (ii) can be written $y_i \succsim_i x_i$, for every $i \in S$ and $y_i \succ_i x_i$ for some $i \in S$. Thus, this standard objection becomes an Aubin objection with rational coefficients.
- A standard counterobjection $(\bar{T}, z)$ to $(\bar{S}, y)$ in a replicated economy $r\mathcal{E}$

is defined by the coalition $\bar{T}$, formed by $1 \leqslant a_i \leqslant r$ consumers of type $i \in T \subset N$ and consumption plans z_{ij} with $i \in T$ and $j \in \{1, \ldots, a_i\}$, such that (i) $\sum_{ij \in \bar{T}} z_{ij} \leqslant \sum_{i \in T} a_i \omega_i$, (ii) $z_{ij} \succ_j y_{ij}$, if consumer $ij \in \bar{T} \cap \bar{S}$ and $z_{ij} \succ_j x_j$ if $ij \in \bar{T} \setminus \bar{S}$. In this counterobjection mechanism we can also consider, as before, that $z_{ij} = z_i$ for all j and thus, the standard counterobjection may be confused with an Aubin counterobjection with rational coefficients.

The formal difference is basically due to the counterobjection process that comes from the consideration of the Aubin's veto mechanism. If an agent of type i belongs to the coalition $\bar{S}$ objecting an allocation rx via y and this objection has a counterobjection $(\bar{T}, z)$, in which another agent ik of the same type is involved, the definition of Aubin bargaining set requires that $z_{ik} \succ_i y_{ik}$, whereas the definition that Mas-Colell states and Anderson *et al.* use for replicas, only requires $z_{ik} \succ_i x_i$ instead[3].

Economically, following Aubin's approach, agents of the same type behave just as an individual. We can think of agents of same type as representatives of a firm, a trade union, a political party or an institution. Then, when an agent of type i belongs to an objecting coalition S obtaining y_i, no other representative is allowed to participate in a counterobjection obtaining some bundle worse than y_i. However, within Mas-Colell's (or Anderson *et al.*) approach, this kind of coordination among agents of the same type is ruled out and they may behave independently in the respective objection and counterobjection mechanisms.

Aubin's objection-counterobjection mechanism is economically relevant when each agent in the finite economy is an institution that has a large enough number of representatives. A model to represent this scenario is the sequence of replicas of the original finite economy. It seems therefore appropriate to investigate further on the asymptotic behavior of such a model.

4.3 About nonconvergence

With the standard veto mechanism that underlies Mas-Colell's definition of bargaining set, Anderson *et al.* (1997) showed the nonconvergence of the sequence of bargaining sets of the replicated economies to the set of Walrasian allocations [4].

[3]Since the difference affects only to the counterobjection process, the notion of core remains unaltered regardless of which notion is used.

[4]Note that Anderson *et al.* nonconvergence result for exchange economies prevents a general convergence result for production economies, in contrast with Liu (2017).

For it, these authors provide a robust example of an exchange economy with two consumers and two commodities in which there are Pareto optimal and individually rational non Walrasian allocations x, such that rx belongs to the bargaining set of the r-replicated economy for every natural number r.

Next, we will address the following question: is it possible to strengthen the definition used by Anderson *et al.* bargaining set for a sequence of replicated economies and get a convergence result? As we will see, without additional requirements, the answer is negative.

For it, let us consider the bargaining set resulting from the previous conception of counterobjections à la Aubin to obtain a notion of bargaining set in a replicated economy $r\mathcal{E}$ (or, more generally, $r\mathcal{E}_P$) that we denote by $B^*(r\mathcal{E})$ (respectively, $B^*(r\mathcal{E}_P)$) and that is defined by the following r-objection and r-counterobjection mechanism:

Definition 4.1 *An allocation x is r-objected if there exist $S \subset N$, an allocation $y = (y_i, i \in S)$ and an integer $1 \leqslant r_i \leqslant r$ for each $i \in S$, with $\sum_{i\in S} r_i y_i \leqslant \sum_{i\in S} r_i \omega_i$ and $y_i \succsim x_i$ for all $i \in S$, with strict preference for some $j \in S$.*

We say that (T, z) r-counterobjects the r-objection (S, y) if $z = (z_i, i \in T \subset N)$, and there exists an integer $1 \leqslant b_i \leqslant r$ for each $i \in T$ such that $\sum_{i\in T} b_i z_i \leqslant \sum_{i\in T} b_i \omega_i$ and $z_i \succ y_i$ for all $i \in S \cap T$, and $z_i \succ x_i$ for all $i \in T \setminus S$.

The bargaining set $B^*(r\mathcal{E})$ is the set of feasible allocations for which there is no r-justified objection.

Note that $B^*(r\mathcal{E})$ is just the "Aubin bargaining set" of $\mathcal{E}$ if the rates of participation are restricted to rational numbers in $(0, 1]$ whose denominators are no larger than r. Observe also that an r-objection is just a standard objection in a replicated economy $r\mathcal{E}$ and thus in any other $r'\mathcal{E}$ with $r' \geqslant r$. However, the requirements for an r-counterobjection are stronger than Mas-Colell's. Next we go further by showing that our bargaining set $B^*(r\mathcal{E})$ is smaller than Mas-Colell's.

Theorem 4.1 *$B^*(r\mathcal{E}) \subset B_M(r\mathcal{E})$ for every replicated economy $r\mathcal{E}$. Therefore $\cap_{r\in\mathbf{N}} B^*(r\mathcal{E}) \subset \cap_{r\in\mathbf{N}} B_M(r\mathcal{E})$.*

Proof. Let x be an allocation such that $(\bar{S}, \bar{y})$ is a justified objection to rx as defined by Mas-Colell, then $\bar{y}$ belongs to the core of the economy restricted to

$\bar{S}$ (see Proposition 3.1). We will show that this fact implies $\bar{y}_{ij} \sim_i \bar{y}_{ik}$ for every $ij, ik \in \bar{S}$. For it, let us assume that for some $kj, kj' \in \bar{S}, \bar{y}_{kj} \succ_k \bar{y}_{kj'}$. Let $io \in \bar{S}$ such that $y_{ij} \succsim_i y_{io}$ for all $ij \in \bar{S}$ and let $S = \{i \in N : r_i > 0\}$, where r_i is the number of members of type i in the coalition $\bar{S}$. The convexity of preferences (see Footnote 2) guarantees that the subcoalition of $\bar{S}$ formed by the individuals who receive the least desired bundle of each type that form part of $\bar{S}$ would block $(\bar{S}, \bar{y})$ via the allocation $z = (z_{io}, i \in S)$ given by $z_{io} = z_i = \frac{1}{r_i} \sum_{j=1}^{r_i} \bar{y}_{ij}$ because $z_i \succsim_i \bar{y}_{io}$ for all $i \in S$ and $z_k \succ_k \bar{y}_{ko}$. In addition, $\sum_{i \in S} z_i = \sum_{i \in S} \frac{1}{r_i} \sum_{j=1}^{r_i} \bar{y}_{ij} \leqslant \sum_{i \in S} \omega_i$. Since $\bar{y}_{ij} \sim \bar{y}_{ik}$ for every $ij, ik \in \bar{S}$, we have $z_i = \frac{1}{r_i} \sum_{j=1}^{r_i} \bar{y}_{ij} \succsim_i \bar{y}_{ij} \succsim_i x_i$ for all i and $z_i \succ_i x_i$ for some i. Thus, (S, z) is an r-objection to x. Since any r-counterobjection to (S, z) is a counterobjection à la Mas-Colell to $(\bar{S}, \bar{y})$, we conclude that (S, z) is an r-justified objection to x.

Q.E.D.

Remark. The case $Y^S = \mathbb{R}^\ell_- = \{y \in \mathbb{R}^\ell : y \leq 0\}$ for every coalition S fulfills the assumptions (P.1) and (P.2) required in Theorem 5.5 in Liu (2017). Thus, this case becomes an economy without production and therefore, the example in Anderson *et al.* (1997) shows the impossibility of convergence. In addition, our Theorem 4.1 shows that our bargaining set is smaller than the one considered in Anderson *et al.* (1997) and even so, there is no convergence.

The previous inclusions are, in general, strict. To see this, check Hervés-Estévez and Moreno-García (2015b and 2017b), where it is shown that for the economy $\mathcal{E}$ in the example by Anderson *et al.* (1997) we have $\cap_{r \in \mathbb{N}} B^*(r\mathcal{E}) = W(\mathcal{E})$, whereas $W(\mathcal{E})$ is strictly contained in $\cap_{r \in \mathbb{N}} B_M(r\mathcal{E})$.

Moreover, the counterexample in Hervés-Estévez and Moreno-García (2017b), shows that even for the smaller bargaining sets $B^*(r\mathcal{E})$, its convergence to the set of Walrasian allocations is not possible without additional assumptions. To be precise, in the economy considered in such a counterexample we find non-Walrasian allocations for which the unique potential objecting coalition which would be able to prevent them to belong to some bargaining set $B^*(r\mathcal{E})$ requires participations rates of consumers given by an irrational number and, thus, there is no justified objection in any replica $r\mathcal{E}$.

4.4 Restricting coalition formation

As we have stressed in the previous paragraphs, the counterobjection mechanism *à la* Aubin has relevant differences with the standard two-step blocking mechanism in replicated economies. This subsection highlights these differences by analyzing how the the corresponding bargaining sets are affected when we restrict the family of coalitions involved in the objection-counterobjection mechanism. Observe that restricting the set of coalitions which are able to object enlarges the core, whereas restricting the coalitions in the counterobjection mechanism makes easier for an objection to become credible or justified. Thus, the overall effect of restricting the coalitions involved in the objection-counterobjection mechanism is unclear.

In the scenario of a continuum economy, Schmeidler (1972) and also Grodal (1972) have shown that it is enough to consider the blocking power of arbitrarily small coalitions (coalitions with measure less than any given threshold) in order to block any non-competitive allocation. However, Schjødt and Sloth (1994) proved that if one restricts the coalitions that can enter into the objection and counter-objection mechanism to those whose measure is arbitrarily small, then the corresponding Mas-Colell's bargaining set becomes strictly larger than the original one.

To see the difference with the Aubin counterobjection process, consider an allocation x in the original finite production economy $\mathcal{E}_P$ and the corresponding step function f_x in the continuum n-types economy $\mathcal{E}_P^c$. Observe that an objection *à la* Aubin (S, y) to x in $\mathcal{E}_P$, with $\sum_{i \in S} \alpha_i (y_i - \omega_i) = \xi \in Y$, can be identified with a standard objection (Mas-Colell objection) $(\hat{S}, f_y)$ to f_x in $\mathcal{E}_P^c$, where $\hat{S} \subset I$ is any coalition such that $\mu(\hat{S} \cap I_i) = \alpha_i$ if i belongs to S and $\mu(\hat{S} \cap I_i) = 0$ if i is not in S.

Let δ-$B(\mathcal{E}_P)$ denote the bargaining set of the economy $\mathcal{E}_P$ where the participation rate of any agent in any coalition, both in the objection and counterobjection procedure, is restricted to be less or equal than δ.

Next theorem contrasts with the result by Schjødt and Sloth (1994) highlighting the differences between Aubin's and Mas-Colell's counterobjection mechanisms. See also Hervés-Estévez and Moreno-García (2015a).

Theorem 4.2 *All the δ-bargaining sets are equal and coincide with the bargaining set in the finite economy $\mathcal{E}$. That is, δ-$B(\mathcal{E}_P) = B(\mathcal{E}_P)$, for every $\delta \in (0, 1]$.*

Proof. Let an allocation y and a coalition S with participation rates $\lambda_i, i \in S$ such that $\sum_{i \in S} \lambda_i (y_i - \omega_i) \in Y$. It suffices to note that there exists $(\alpha_i, i \in S)$, with $\alpha_i \in (0, \delta]$ for every $i \in S$ such that $\sum_{i \in S} \alpha_i (y_i - \omega_i) \in Y$. To see this, let M be large enough so that $\alpha_i = \lambda_i / M \leqslant \delta$, for every $i \in S$. The same reasoning holds for the case of both objections and counterobjections.

Q.E.D.

Symmetrically to Schmeidler's (1972) and Grodal's (1972) core characterizations for atomless economies, Vind (1972) showed that in order to block any non-competitive allocation it is enough to consider the veto power of arbitrarily large coalitions. Hervés-Estévez and Moreno-García (2017a) show that, for echange economies, such restriction does not produce a similar effect for Aubin's bargaining set, neither in the objection (Example 1) nor in the counterobjection (Example 2).

Finally, we remark that in the case of a continuum economy $\mathcal{E}^c$, the objection-counterobjection mechanism we define in this paper is relevant only when there is a positive measure set of identical consumers. Otherwise, it is exactly the same definition as in Mas-Colell (1989). This is due to the fact that our objection-counterobjection mechanism sets an additional restriction on the counterobjection when identical agents participate both in the objection and in the counterobjection. Then, if $\hat{B}^*(\mathcal{E}^c)$ denotes our bargaining set, we always have that $B^*(\mathcal{E}^c) \subset B_M(\mathcal{E}^c)$. Now, given the equivalence of Mas-Colell's bargaining set and competitive allocations for continuum economies, we have $B^*(\mathcal{E}^c) = B_M(\mathcal{E}^c)$. In particular, if one considers a m-types continuum economy $\mathcal{E}^c_m$ –where the mechanism we define is relevant– one still has $B^*(\mathcal{E}^c_m) = B_M(\mathcal{E}^c_m)$.

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