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Title:

THE ADJUSTED PROPORTIONAL AND THE MINIMAL OVERLAP RULES RESTRICTED TO THE LOWER-HALF, HIGHER-HALF, AND MIDDLE DOMAINS

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The adjusted proportional and the minimal overlap rules restricted to the lower-half, higher-half, and middle domains

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Abstract

We analyze whether or not some properties (order preservation under claims variations, order preservation under population variation, progressivity and regressivity) that are violated by the adjusted proportional and the minimal overlap rules on the domain of claims problems are nevertheless satisfied by these rules on some relevant subdomains. Lorenz-based characterizations of the adjusted proportional rule restricted to the lower-half and higher-half domains are given. We also prove that the adjusted proportional rule Lorenz-dominates the minimal overlap rule. Finally, we summarize the Lorenz-dominance ranking of ten rules for claims problems on the lower-half, higher-half, and middle domains.

Keywords: Claims problems, adjusted proportional rule, minimal overlap rule, Lorenz-domination.

1 Introduction

A claims problem arises when an amount has to be divided among a set of agents with claims that, in aggregate, exceed what is available. A rule is a way of selecting a division among the claimants. The definition of rules and the study of different approaches to evaluate and compare them started with O'Neill (1982), and has produced ever since a vast literature. The model has many applications that include bankruptcy problems, taxation systems, rationing, or the distribution of the carbon budget. For a thorough survey on this subject see Thomson (2019).

The best-known rule is the proportional rule that simply shares the scarce resource proportional to claims. This paper focusses mainly in two rules. The adjusted proportional rule was defined and studied by Curiel et al. (1987). This rule first allocates to each claimant his minimal right, the part of the amount that is left after each other individual is fully compensated. Each agent's claim is revised down to the minimum of the remainder and the difference between his initial claim and his minimal right. Finally, the resulting problem is solved using the proportional rule. In the 12th century, the talmudic scholar Ibn Ezra described a problem consisting in dividing an estate among four sons. The recommendation that he presented was a particular case of a method proposed by Rabad, also in the 12th century, defined for problems such that no claim exceeds the estate. This incompletely specified rule was extended for an arbitrary claims problem by O'Neill (1982), and named the minimal overlap rule by Thomson (2003). Imagine that the amount available consists of distinct parts, and that each agent, instead of expressing his claim in some abstract way, claims specific parts of total amount equal to his claim. The minimal overlap rule chooses awards vectors that minimize "extent of conflict" over each unit available. Alcalde et al. (2005, 2008); Chun and Thomson (2005); Hendrickx et al. (2007) have given implicit formulae and new representations and interpretations of the minimal overlap rule.

But the inventory of rules is rich. In this paper, we also considered: the constrained equal awards rule, the constrained equal losses rule, the Piniles's rule, the Talmud rule, the constrained egalitarian rule, the random arrival rule, and the average of awards rule. A rule might be selected by the appeal of its own definition and by the properties that it satisfies or violates. In fact, a rule might be characterized as the only

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one that satisfies certain properties, or axioms. When a rule violates a property, it is relevant to know if the rule satisfies it when restricted to a subdomain of problems. Some meaningful subdomains have already received attention. Aumann and Maschler (1985) argue that the half sum of the claims is an important point (literally, a watershed). In fact, the definition of the Talmud rule, for example, depends on whether or not the endowment is lower or bigger that the half sum of the claims. These sets of problems are called the lower-half and higher-half domains. Thomson (2019) discusses the domain of simple claims problems, those for which each claim is at most as large as the endowment. Mirás Calvo et al. (2020) study the set of award vectors for problems such that the largest claim is bigger that the half sum of claims. In that case, they show that for each problem for which the endowment differs from the half sum of claims less than the largest claim, the set of awards vectors has a common simple structure. These problems are part of the middle domain. We show that, restricted to this domain, the Talmud, the random arrival, the adjusted proportional and the average of awards rules coincide.

Our first goal in this paper is to analyze if the adjusted proportional and the minimal overlap rules satisfy the following properties either on the general domain of claims problems or restricted to any of the above mentioned subdomains: other-regarding claim monotonicity, order preservation under claims variations, progressivity, regressivity, population monotonicity, and order preservation under population variation. In particular, we show that the adjusted proportional rule violates order preservation under claims variations.

Rules can be compared and ranked. The Lorenz criterion is widely used for this purpose. In order to compare a pair of awards vectors, rearrange the coordinates of each vector in a non-decreasing order. Then one vector Lorenz-dominates the other if the first coordinate and all the cumulative sums of the rearranged coordinates are greater with the former than with the latter. Bosmans and Lauwers (2011) provide an exhaustive ranking of nine of the rules mentioned above (all but the average of awards rule that was introduced by Mirás Calvo et al. (2020)). Their work covers and extends many related contributions on the ranking of rules. In addition, they characterize six of them (the constrained egalitarian rule, the constrained equal awards rule, the Piniles' rule, the minimal overlap rule, and the Talmud rule) as maximal or minimal with respect to the Lorenz-dominance relation. One important aspect of their analysis is that, since some rules are not Lorenz-comparable, they restrict the comparison to the lower-half and higher-half domains. Naturally, on the restricted domains the ranking of rules is richer.

This paper has a twofold objective. On the one hand, the Lorenz-comparison of the adjusted proportional and the minimal overlap rules presented in Bosmans and Lauwers (2011) relies on a characterization of the minimal overlap rule as Lorenz-minimal in the class of rules satisfying order preservation, $\frac{1}{|N|}$-truncated-claims lower bounds on awards, null claims consistency, and order preservation under claims variations. But, as we already mentioned, and contrary to what is stated in Bosmans and Lauwers (2011), the adjusted proportional rule violates order preservation under claims variations and, consequently, one can not apply the characterization of the minimal overlap rule to rank both rules. We show that, indeed, the adjusted proportional rule Lorenz-dominates the minimal overlap rule. We derive the result by providing a new Lorenz-based characterization of the adjusted proportional rule on the lower-half and higher-half domains. Progressivity and regressivity are the key properties. On the other hand, we refine the ranking of rules restricted to the lower-half and higher-half domains by adding the middle domain. We incorporate the average of awards rule to the picture. In fact, Mirás Calvo et al. (2021) compare it with the adjusted proportional rule by means of the Lorenz-based characterization obtained here. Figure 6 illustrates and summarizes the ranking of the ten rules on the four subdomains.

In Section 2 we introduce the basic definitions, notations and properties. Some relevant domains of claims problems are presented in Section 3. We show in Section 4 that the adjusted proportional rule fails others-regarding claim monotonicity and order preservation under claims variations but satisfies both properties when restricted to the lower-half domain. Progressivity and regressivity of the adjusted proportional and the minimal overlap rules are analyzed in Section 5. We establish that the adjusted proportional rule Lorenz-dominates the minimal overlap rule by providing a Lorenz-based characterizations of the adjusted proportional rule restricted to the lower-half and higher-half domains in Section 7. Finally, Section 8 summarizes the results.

2 Preliminaries

Let $\mathcal{N}$ be the set of all finite subsets of the natural numbers $\mathbb{N}$. Given $N \in \mathcal{N}$, $x \in \mathbb{R}^N$, and $S \in 2^N$ let $|N|$ be the number of elements of N and $x(S) = \sum_{i \in S} x_i$. If $N' \subset N \in \mathcal{N}$ and $x \in \mathbb{R}^N$, let $x_{N'} = \left(x_i\right)_{i \in N'} \in \mathbb{R}^{N'}$ be the projection of x onto $\mathbb{R}^{N'}$. In particular denote $x_{-i} = x_{N\setminus\{i\}} \in \mathbb{R}^{N\setminus\{i\}}$ the vector obtained by neglecting the ith-coordinate of x, i.e., $x_{-i} = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$. For simplicity, we will write $x = (x_{-i}, x_i)$.

A claims problem with set of claimants $N \in \mathcal{N}$ is a pair (E, d) where $E \geq 0$ is the endowment to be divided and $d \in \mathbb{R}^N$ is the vector of claims satisfying $d_i \geq 0$ for all $i \in N$ and $d(N) \geq E$. We denote the class of claims problems with set of players N by B^N.

For each $(E, d) \in B^N$ and each $i \in N$ let $D_{-i} = d(N) - d_i = d(N\setminus\{i\})$. The minimal right and truncated claim of claimant $i \in N$ in $(E, d) \in B^N$ are the quantities $m_i(E, d) = \max\{0, E - D_{-i}\}$ and $t_i(E, d) = \min\{E, d_i\}$, respectively. Let $m(E, d) = \big(m_i(E, d)\big)_{i\in N}$ and $t(E, d) = \big(t_i(E, d)\big)_{i\in N}$. Let us write $t = t(E, d)$ and $m = m(E, d)$ if no confusion is possible.

Let $\mathbb{R}^n_{\leq}$ be the set of nonnegative n-dimensional vectors $x = (x_1, \ldots, x_n)$ with coordinates ordered from small to large, i.e., $0 \leq x_1 \leq \ldots \leq x_n$. For simplicity, given $(E, d) \in B^N$ with $|N| = n$, we will assume throughout the paper that $N = \{1, \ldots, n\}$ and that $d \in \mathbb{R}^n_{\leq}$. As a consequence of such an arrangement of the claims we have that $d_i \leq D_{-i}$, $D_{-i} \geq D_{-(i+1)}$ and $m_i \leq m_{i+1}$ for all $i \in N\setminus\{n\}$. Nevertheless, as it is illustrated in Figure 1, we can have either $d_n \leq D_{-n}$ or $D_{-n} \leq d_n$. In any case, $\frac{1}{2}d(N)$ is the middle point of the line segment with endpoints d_n and D_{-n}. In fact, $\frac{1}{2}d(N)$ is also the middle point of the intervals $[d_i, D_{-i}]$ for all $i \in N\setminus\{n\}$.

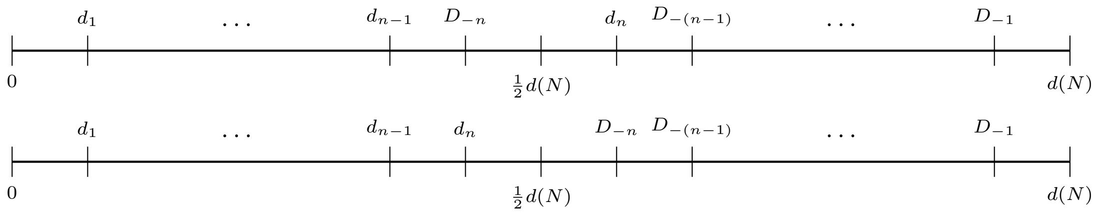

Figure 1: Claims arranged in ascending order on the interval $[0, d(N)]$.

A vector $x \in \mathbb{R}^N$ is an awards vector of $(E, d) \in B^N$ if $0 \leq x_i \leq d_i$ for all $i \in N$ and $x(N) = E$. Let $X(E, d)$ be the set of awards vectors for $(E, d) \in B^N$. A rule is a function $\mathcal{R}\colon B^N \to \mathbb{R}^N$ assigning to each claims problem $(E, d) \in B^N$ an awards vector $\mathcal{R}(E, d) \in X(E, d)$, that is, a way of associating with each claims problem a division between the claimants of the amount available. For example, if $N = \{1, 2\}$ and $(E, (d_1, d_2)) \in B^N$, the concede-and-divide rule (CD) is given by:

$$\mathrm{CD}(E, (d_1, d_2)) = \begin{cases} \big(\frac{E}{2}, \frac{E}{2}\big) & \text{if } 0 \leq E \leq d_1 \\ \big(\frac{d_1}{2}, E - \frac{d_1}{2}\big) & \text{if } d_1 \leq E \leq d_2 \\ \big(\frac{E+d_1-d_2}{2}, \frac{E-d_1+d_2}{2}\big) & \text{if } d_2 \leq E \leq d_1 + d_2 \end{cases}.$$

A rule $\mathcal{R}$ satisfies:

- *anonymity* if for each $(E, d) \in B^N$, each $\pi \in \Pi^N$, and each $i \in N$, we have $\mathcal{R}_{\pi(i)}\big(E, (d_{\pi(i)})\big) = \mathcal{R}_i(E, d)$, where Π^N is the class of bijections from N into itself.
- *the midpoint property* if $\mathcal{R}(\frac{1}{2}d(N), d) = \frac{d}{2}$.
- *self-duality* if for each $(E, d) \in B^N$ we have $\mathcal{R}(E, d) = d - \mathcal{R}\big(d(N) - E, d\big)$.
- *minimal rights first* if for each $(E, d) \in B^N$ we have $\mathcal{R}(E, d) = m(E, d) + \mathcal{R}\big(E - \sum_{i\in N} m_i(E, d), d - m(E, d)\big)$.
- *claims truncation invariance* if for each $(E, d) \in B^N$ we have $\mathcal{R}(E, d) = \mathcal{R}(E, t(E, d))$.
- *order preservation in awards* if for each $(E, d) \in B^N$ we have $\mathcal{R}_i(E, d) \leq \mathcal{R}_{i+1}(E, d)$ for all $i \in N\setminus\{n\}$.
- *order preservation in losses* if for each $(E, d) \in B^N$ we have $d_i - \mathcal{R}_i(E, d) \leq d_{i+1} - \mathcal{R}_{i+1}(E, d)$ for all $i \in N\setminus\{n\}$.
- $\frac{1}{|N|}$*-truncated-claims lower bounds on awards*, if for each $(E, d) \in B^N$ then $\mathcal{R}(E, d) \geq \frac{1}{|N|}t(E, d)$.

- *endowment monotonicity* if for each $(E, d) \in B^N$ and each $E' \geq E$, if $d(N) \geq E' \geq E$ we have $\mathcal{R}(E', d) \geq \mathcal{R}(E, d)$.

With each rule $\mathcal{R}$ we can associate a unique dual rule $\mathcal{R}^*$, defined by $\mathcal{R}^*(E, d) = d - \mathcal{R}\big(d(N) - E, d\big)$. Two properties are dual if, whenever a rule satisfies one of them, its dual satisfies the other. A property is self-dual if it coincides with its dual. The claims problems $(E, d) \in B^N$ and $(d(N) - E, d) \in B^N$ are dual claims problems. Given a domain of claims problems $\Omega \subset B^N$ the domain of dual claims problems $\Omega^* = \{(d(N) - E, d) \in B^N : (E, d) \in \Omega\}$ is the dual domain of Ω. Clearly, if a rule $\mathcal{R}$ satisfies a property when restricted to a domain $\Omega \subset B^N$ then its dual rule $\mathcal{R}^*$ satisfies the dual property on the dual domain Ω^*.

Let us introduce the three rules that are the main focus of our analysis:

- The proportional rule, PRO, is given, for each $(E, d) \in B^N$ and each $i \in N$, by $\mathrm{PRO}_i(E, d) = \frac{d_i}{d(N)} E$.
- The adjusted proportional rule, APRO, is defined, for each $(E, d) \in B^N$ and each $i \in N$, as:

$$\mathrm{APRO}_i(E, d) = m_i + \mathrm{PRO}_i\big(E - M, (\min\{d_j - m_j, E - M\})_{j \in N}\big),$$

  where $M = \sum\limits_{i \in N} m_i$.

- Let $d_0 = 0$. The minimal overlap rule, MO, is defined, for each $(E, d) \in B^N$ and each $i \in N$, as:
  i) If $E \leq d_n$ then $\mathrm{MO}_i(E, d) = \frac{t_1}{n} + \frac{t_2 - t_1}{n-1} + \cdots + \frac{t_i - t_{i-1}}{n-i+1}$.
  ii) If $E > d_n$, let $s^* \in (d_{k*}, d_{k^*+1}]$, with $k^* \in \{0, 1, \ldots, n-2\}$, be the unique solution to the equation $\sum\limits_{i \in N} \max\{d_i - s, 0\} = E - s$. Then,

$$\mathrm{MO}_i(E, d) = \begin{cases} \frac{d_1}{n} + \frac{d_2 - d_1}{n-1} + \cdots + \frac{d_i - d_{i-1}}{n-i+1} & \text{if } i \in \{1, \ldots, k^*\} \\ \mathrm{MO}_i(s^*, d) + d_i - s^* & \text{if } i \in \{k^* + 1, \ldots n\} \end{cases}.$$

In addition to the three rules already defined, throughout this paper other basic rules will be considered: the constrained equal awards rule (CEA), the constrained equal losses rule (CEL), the constrained egalitarian rule (CE), the Piniles‘s rule (PIN), the Talmud rule (T), the random arrival rule (RA), and the average of awards rule (AA). Mirás Calvo et al. (2020) define for each claims problem the average of awards rule as the geometric center (the centroid) of the set of awards vectors.[1] The precise definitions of the other rules can be found, for instance, in Thomson (2019). Table 1, adapted from Thomson (2019), summarizes which of the above properties are satisfied by these rules. A check mark, ✓, in a cell means that the property in the row is satisfied by the rule indexing the column. A minus sing, −, means the opposite. We have assumed

	PRO	APRO	MO	CEA	CEL	CE	PIN	T	RA	AA
Anonymity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Midpoint	✓	✓	−	−	−	✓	✓	✓	✓	✓
Self-duality	✓	✓	−	−	−	−	−	✓	✓	✓
Minimal rights first	−	✓	✓	−	✓	−	−	✓	✓	✓
Claims truncation invariance	−	✓	✓	✓	−	−	−	✓	✓	✓
Order preservation	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
Endowment monotonicity	✓	✓	✓	✓	✓	✓	✓	✓	✓	✓
$\frac{1}{\lvert N \rvert}$-truncated-claims lower bounds	−	✓	✓	✓	−	✓	✓	✓	✓	✓

Table 1: Main properties satisfied by the ten rules.

that given a claims problem $(E, d) \in B^N$ the coordinates of the vector of claims are ordered from small to large, that is, $d \in \mathbb{R}^n_{\leq}$. Since the ten rules that we discuss in this paper are anonymous this assumption is fully justified. Moreover, if $\mathcal{R}$ is a rule that satisfies order preservation in awards, as happens with our ten rules, then $\mathcal{R}(E, d) \in \mathbb{R}^n_{\leq}$.

[1]The average of awards rule is also called the core-center rule.

3 Subdomains of claims problems

Sometimes a rule violates a property but satisfies it when restricted to a particular subclass of claims problems. Let us introduce some subdomains of claims problems that are particularly relevant.

The subdomain of claims problems for which the endowment is lower than the half-sum of claims, $B_L^N = \{(E,d) \in B^N : E \leq \frac{1}{2}d(N)\}$, is called the lower-half domain. The subdomain of claims problems for which the endowment is bigger than the half-sum of claims, $B_H^N = \{(E,d) \in B^N : E \geq \frac{1}{2}d(N)\}$, is called the higher-half domain. Naturally, the lower-half domain and the higher-half domain are dual. Let us called the intersection $B_L^N \cap B_H^N = \{(E,d) \in B^N : E = \frac{1}{2}d(N)\}$ the midpoint domain: the class of claims problems for which the amount to divide is exactly the half-sum of the claims. Obviously, a rule $\mathcal{R}$ satisfies the midpoint property if $\mathcal{R}(E,d) = \frac{d}{2}$ for all $(E,d) \in B_L^N \cap B_H^N$. It is worth noting that if $(E,d) \in B_L^N$ then $\mathrm{T}(E,d) = \mathrm{CE}(E,d) = \mathrm{PIN}(E,d)$.

As we have already pointed out, see Figure 1, given a claims vector $d \in \mathbb{R}^n_{\leq}$, then $\frac{1}{2}d(N)$ is the middle point of the line segment with endpoints d_n and D_{-n}, but we can have both $d_n \leq D_{-n}$ or $D_{-n} \leq d_n$. Mirás Calvo et al. (2020) show that the set of awards $X(E,d)$ has a particularly simple structure for all claims problems $(E,d) \in B^N$ such that $D_{-n} \leq E \leq d_n$. In fact, the projection of the set of awards onto $\mathbb{R}^{N\setminus\{n\}}$ is a $(n-1)$-rectangle that does not depend on the endowment. To be precise, given $(E,d) \in B^N$ such that $D_{-n} \leq E \leq d_n$ if $\pi \colon \mathbb{R}^N \to \mathbb{R}^{N\setminus\{n\}}$ is the projection $\pi(x) = x_{-n} \in \mathbb{R}^{N\setminus\{n\}}$ then $\pi(X(E,d)) = \prod_{i=1}^{n-1}[0,d_i]$. The class of claims problems $B_M^N = \big\{(E,d) \in B^N \colon \min\{D_{-n}, \frac{1}{2}d(N)\} \leq E \leq \max\{d_n, \frac{1}{2}d(N)\}\big\}$ is called the middle domain.

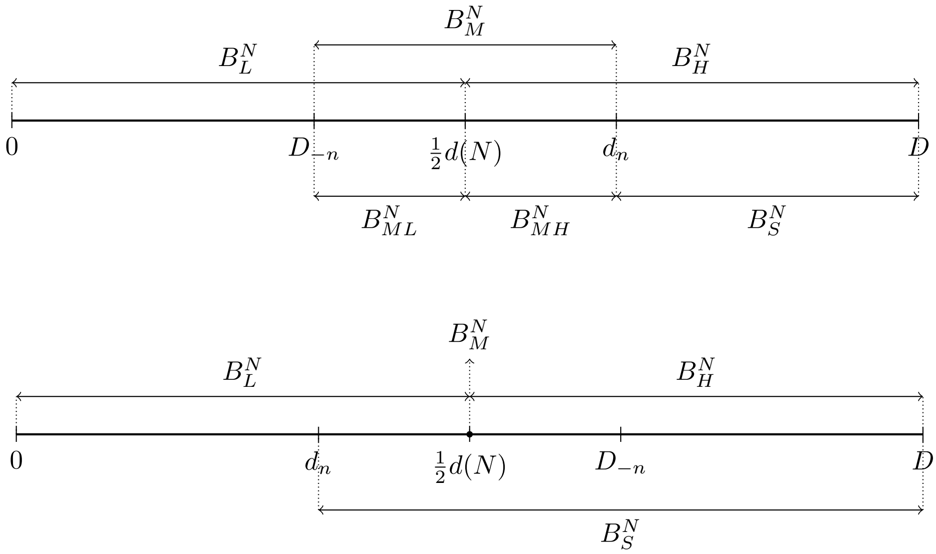

Figure 2: The subdomains of claims problems relevant in our study.

The intersections of the middle domain with the lower-half and higher-half domains are denoted by:

$$B_{ML}^N = B_M^N \cap B_L^N = \big\{(E,d) \in B^N \colon \min\{D_{-n}, \tfrac{1}{2}d(N)\} \leq E \leq \tfrac{1}{2}d(N)\big\}$$
$$B_{MH}^N = B_M^N \cap B_H^N = \big\{(E,d) \in B^N \colon \tfrac{1}{2}d(N) \leq E \leq \max\{d_n, \tfrac{1}{2}d(N)\}\big\}.$$

Clearly, B_{ML}^N and B_{MH}^N are dual domains.

If a claims problem belongs to the middle domain, $(E,d) \in B_M^N$, then a natural division is the one that gives to all the claimants, except the last, the geometric center of the $(n-1)$-rectangle $\prod_{i=1}^{n-1}[0,d_i]$, and to the last claimant what is left, i.e., the division given by the average of awards rule:

$$\mathrm{AA}(E,d) = \left(\tfrac{d_1}{2}, \ldots, \tfrac{d_{n-1}}{2}, E - \tfrac{D_{-n}}{2}\right) \in X(E,d).$$

For claims problems with just two claimants ($|N| = 2$), the middle domain is the set $B_M^N = \{(E,(d_1,d_2)) \in B^N \colon d_1 \leq E \leq d_2\}$. Therefore, if $(E,d) \in B_M^N$ then $AA(E,d) = \left(\frac{d_1}{2}, E - \frac{d_1}{2}\right)$. In fact, the average of awards rule coincides with the concede-and-divide rule for two claimants. Recall that the concede-and-divide rule is the only two-claimant rule satisfying claims truncation invariance, minimal rights first, and the midpoint property. Mirroring this characterization, we can identify rules that, restricted to the middle domain, coincide with the average of awards division.

Proposition 1. *Let $\mathcal{R}$ be a rule. Then:*

1. *If $\mathcal{R}$ satisfies claims truncation invariance and endowment monotonicity on B^N_{ML}, and the midpoint property then $\mathcal{R}(E,d) = \mathrm{AA}(E,d)$ for all $(E,d) \in B^N_{ML}$.*
2. *If $\mathcal{R}$ satisfies claims truncation invariance and minimal rights first on B^N_{ML}, and the midpoint property then $\mathcal{R}(E,d) = \mathrm{AA}(E,d)$ for all $(E,d) \in B^N_{ML}$.*
3. *If $\mathcal{R}$ satisfies minimal rights first and endowment monotonicity on B^N_{MH}, and the midpoint property then $\mathcal{R}(E,d) = \mathrm{AA}(E,d)$ for all $(E,d) \in B^N_{MH}$.*
4. *If $\mathcal{R}$ satisfies claims truncation invariance and minimal rights first on B^N_{MH}, and the midpoint property then $\mathcal{R}(E,d) = \mathrm{AA}(E,d)$ for all $(E,d) \in B^N_{MH}$.*

Proof. Let $\mathcal{R}$ be a rule that satisfies claims truncation invariance and endowment monotonicity on B^N_{ML}, and also the midpoint property. Take $(E,d) \in B^N_{ML}$. If $\min\{D_{-n}, \frac{1}{2}d(N)\} = \frac{1}{2}d(N)$ then $E = \frac{1}{2}d(N)$, and, by the midpoint property, $\mathcal{R}(\frac{1}{2}d(N), d) = \mathrm{AA}(\frac{1}{2}d(N), d) = \frac{d}{2}$. On the other hand, if $\min\{D_{-n}, \frac{1}{2}d(N)\} = D_{-n}$ then $E \in [D_{-n}, \frac{1}{2}d(N)]$ and $E \leq d_n$. By claims truncation invariance and the midpoint property, we have that

$$\begin{aligned} \mathcal{R}(D_{-n}, d) &= \mathcal{R}(D_{-n}, (d_{-n}, D_{-n}) = \left(\tfrac{d_1}{2}, \ldots, \tfrac{d_{n-1}}{2}, \tfrac{D_{-n}}{2}\right) \\ \mathcal{R}(\tfrac{1}{2}d(N), d) &= \left(\tfrac{d_1}{2}, \ldots, \tfrac{d_{n-1}}{2}, \tfrac{d_n}{2}\right). \end{aligned}$$

But, since $\mathcal{R}$ satisfies endowment monotonicity on B^N_{ML}, for each $E \in (D_{-n}, \frac{1}{2}d(N))$ and each $j \in N\backslash\{n\}$, we have $\mathcal{R}_j(D_{-n}, d) \leq \mathcal{R}_j(E,d) \leq \mathcal{R}_j(\frac{1}{2}d(N), d)$, so, necessarily, $\mathcal{R}_j(E,d) = \frac{d_j}{2}$. Therefore, $\mathcal{R}(E,d) = \mathrm{AA}(E,d)$.

Assume now that $\mathcal{R}$ is a rule that satisfies claims truncation invariance and minimal rights first on B^N_{ML}, and the midpoint property. Let $(E,d) \in B^N_{ML}$. When $\min\{D_{-n}, \frac{1}{2}d(N)\} = \frac{1}{2}d(N)$, the result is obvious. If $\min\{D_{-n}, \frac{1}{2}d(N)\} = D_{-n}$ then $E \in [D_{-n}, \frac{1}{2}d(N)]$ and $m(E,d) = (0, \ldots, 0, E - D_{-n})$. By claims truncation invariance, minimal rights first, and the midpoint property,

$$\mathcal{R}(E,d) = \mathcal{R}(E,t) = m + \mathcal{R}(D_{-n}, (d_{-n}, D_{-n})) = (0, \ldots, 0, E - D_{-n}) + \left(\tfrac{d_1}{2}, \ldots, \tfrac{d_{n-1}}{2}, \tfrac{D_{-n}}{2}\right) = \mathrm{AA}(E,d).$$

Finally, since claims truncation invariance and minimal rights first are dual properties and B^N_{ML} and B^N_{MH} are dual domains, the last two statements follow at once. □

The Talmud rule, the random arrival rule, and the adjusted proportional rule satisfy all the properties included in Proposition 1 on the whole domain of claims problems. So, for each claims problem that belongs to the middle domain these rules coincide with the division proposed by the average of awards rule.

Corollary 1. *If $(E,d) \in B^N_M$ then* $\mathrm{T}(E,d) = \mathrm{RA}(E,d) = \mathrm{APRO}(E,d) = \mathrm{AA}(E,d)$*. Moreover, if $(E,d) \in B^N_{ML}$ then* $\mathrm{CE}(E,d) = \mathrm{PIN}(E,d) = \mathrm{T}(E,d) = \mathrm{RA}(E,d) = \mathrm{APRO}(E,d) = \mathrm{AA}(E,d)$.

Proof. The Talmud rule, the random arrival rule, and the adjusted proportional rule satisfy the midpoint property, claims truncation invariance, minimal rights first and endowment monotonicity on the domain of claims games. Therefore, by Proposition 1, $\mathrm{T}(E,d) = \mathrm{RA}(E,d) = \mathrm{APRO}(E,d) = \mathrm{AA}(E,d)$ for all $(E,d) \in B^N_M$. But if $(E,d) \in B^N_{ML}$ then, by definition, $\mathrm{CE}(E,d) = \mathrm{PIN}(E,d) = \mathrm{T}(E,d)$. □

Thomson (2019) discusses another important domain: claims problems in which no claim exceeds the endowment. A claims problem $(E,d) \in B^N$ is a simple claims problem if $E \geq d_i$ for all $i \in N$. Let us denote the domain of simple claims problems by $B^N_S = \{(E,d) \in B^N : E \geq d_i \text{ for all } i \in N\}$. Given a vector of claims $d \in \mathbb{R}^n_{\leq}$, Figure 2 shows schematically the intervals in which the endowment E has to be so that the claims problem (E,d) belongs to each of the subdomains that we have defined.

4 Properties violated by the adjusted proportional rule

We devote this Section to analyze several properties violated by the adjusted proportional rule on the domain of claims problems that, nevertheless, are satisfied by this rule when restricted to the lower-half domain. We start by providing an expression of the adjusted proportional rule on the lower-half domain.

Proposition 2. *Let $(E,d) \in B_L^N$ and $d_0 = 0$. We have that*

1. *If $E \leq \min\{D_{-n}, \frac{1}{2}d(N)\}$, let $k_0 = |\{k \in N : d_k \leq E\}|$ and $\mathcal{T} = \sum_{s=0}^{k_0} d_s + (n-k_0)E$. Then*

$$\mathrm{APRO}_j(E,d) = \mathrm{PRO}_j(E, t(E,d)) = \begin{cases} \frac{d_j E}{\mathcal{T}} & \text{if } j \leq k_0 \\ \\ \frac{E^2}{\mathcal{T}} & \text{if } j > k_0 \end{cases}.$$

2. *If $E \in [D_{-n}, \frac{1}{2}d(N)]$ then $\mathrm{APRO}(E,d) = \left(\frac{d_1}{2}, \ldots, \frac{d_{n-1}}{2}, E - \frac{D_{-n}}{2}\right)$.*

Proof. Let $(E,d) \in B_L^N$. If $(E,d) \in B_{ML}^N$ then, by Corollary 1, $\mathrm{APRO}(E,d) = \mathrm{AA}(E,d)$. Now, if $E \leq \min\{D_{-n}, \frac{1}{2}d(N)\}$ then $m_j = 0$ for all $j \in N$ and $\mathrm{APRO}(E,d) = \mathrm{PRO}(E, t(E,d))$. Observe that, if $k_0 = |\{k \in N : d_k \leq E\}|$ then $\mathcal{T} = \sum_{i \in N} t_i(E,d) = \sum_{s=0}^{k_0} d_s + (n-k_0)E$. The result in this case is straightforward. □

We recall three properties that consider when agent i's claim increases. Claim monotonicity says that agent i's award should not decrease. Other-regarding claim monotonicity requires each of the other claimants to receive at most as much as initially. If there are at least three claimants, order preservation under claims variations says that given any two claimants whose claim remains the same, the change in the award to the smaller one should be at most as large as the change in the award to the larger one. Formally, a rule $\mathcal{R}$ satisfies:

- *claim monotonicity* if for each $(E,d) \in B^N$, each $i \in N$, and each $d_i' > d_i$, then $\mathcal{R}_i(E,(d_{-i},d_i')) \geq \mathcal{R}_i(E,d)$.
- *other-regarding claim monotonicity* if for each $(E,d) \in B^N$, each $i \in N$, and each $d_i' > d_i$, then $\mathcal{R}_j(E,(d_{-i},d_i')) \leq \mathcal{R}_j(E,d)$ for all $j \in N \backslash \{i\}$.
- *order preservation under claims variations* if for each $(E,d) \in B^N$ with $|N| \geq 3$, each $i \in N$, each $d_i' > d_i$, and each pair $\{j,k\} \subset N \backslash \{i\}$ such that $d_j \leq d_k$, then $\mathcal{R}_j(E,d) - \mathcal{R}_j(E,(d_{-i},d_i')) \leq \mathcal{R}_k(E,d) - \mathcal{R}_k(E,(d_{-i},d_i'))$.

It is known, see Thomson (2019) and Mirás Calvo et al. (2020), that the ten rules mentioned in this paper satisfy claim monotonicity. But, contrary to what Bosmans and Lauwers (2011) assert, we show next that the adjusted proportional rule fails both other-regarding claim monotonicity and order preservation under claims variations.

Proposition 3. *The adjusted proportional rule satisfies neither other-regarding claim monotonicity nor order preservation under claims variations.*

Proof. We provide a counterexample. Let $N = \{1,2,3,4,5\}$, $E = 17$, $d = (1,2,3,8,10)$, and $d' = (1,2,4,8,10)$. Clearly, $(E,d) \in B^N$, $(E,d') \in B^N$, $m(E,d) = (0,0,0,1,3)$, and $m(E,d') = (0,0,0,0,2)$, so

$$\mathrm{APRO}(E,d) = (0,0,0,1,3) + \mathrm{PRO}(13,(1,2,3,7,7)) = \left(\tfrac{13}{20}, \tfrac{26}{20}, \tfrac{39}{20}, \tfrac{111}{20}, \tfrac{151}{20}\right)$$
$$\mathrm{APRO}(E,d') = (0,0,0,0,2) + \mathrm{PRO}(15,(1,2,4,8,8)) = \left(\tfrac{15}{23}, \tfrac{30}{23}, \tfrac{60}{23}, \tfrac{120}{23}, \tfrac{166}{23}\right).$$

Since $\mathrm{APRO}_1(E,d') > \mathrm{APRO}_1(E,d)$, the adjusted proportional rule violates other-regarding claim monotonicity. Moreover,

$$\mathrm{APRO}_1(E,d) - \mathrm{APRO}_1(E,d') = -\tfrac{1}{460} > -\tfrac{1}{230} = \mathrm{APRO}_2(E,d) - \mathrm{APRO}_2(E,d').$$

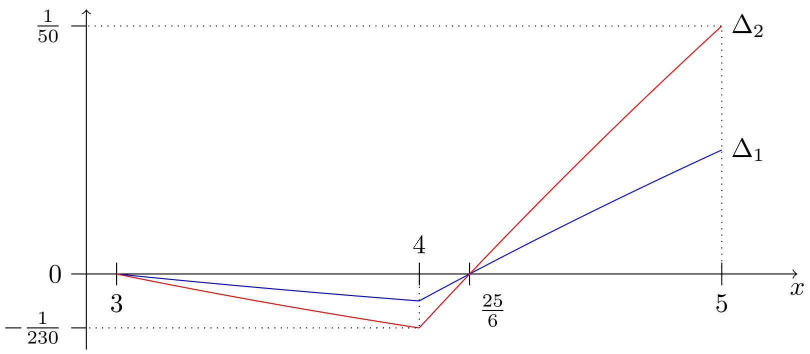

Figure 3: The increments Δ_1 and Δ_2 for the problems $(17, d_x) \in B^N$ with $d_x = (1, 2, x, 8, 10)$, $x \in [3, 5]$.

Therefore, order preservation under claims variation is also violated by the adjusted proportional rule. Figure 3 illustrates that both properties are violated by the adjusted proportional rule if one considers the vector of claims $d' = d_x = (1, 2, x, 8, 10)$, with $x \in [3, 5]$, by comparing the increments $\Delta_i(x) = \text{APRO}_i(E, d) - \text{APRO}_i(E, d_x)$, for $i = 1, 2$. □

Later, as a consequence of Proposition 9 and Example 2, we present an alternative proof that the adjusted proportional rule fails other-regarding claim monotonicity. Now, we establish that, when restricted to B^N_L, the adjusted proportional rule satisfies both other-regarding claim monotonicity and order preservation under claims variations.

Proposition 4. *The adjusted proportional rule rule satisfies both other-regarding claim monotonicity and order preservation under claims variations on the lower-half domain.*

Proof. Let $(E, d) \in B^N_L$, $i \in N$, $d'_i > d_i$, $d' = (d_{-i}, d'_i)$, and denote $\Delta_j = \text{APRO}_j(E, d) - \text{APRO}_j(E, d')$ for $j \in N \backslash \{i\}$. We have to prove that $0 \leq \Delta_j \leq \Delta_k$ for all $\{j, k\} \subset N \backslash \{i\}$ with $d_j \leq d_k$. Since the APRO rule satisfies anonymity, it is sufficient to prove the result when $d_i < d'_i \leq d_{i+1}$. Now, since $(E, d) \in B^N_L$ then $(E, d') \in B^N_L$ and $t_j = t'_j$ for $j \in N \backslash \{i\}$. Let $\mathcal{T} = \sum_{k \in N} t_k$ and $\mathcal{T}' = \sum_{k \in N} t'_k$. If $E \leq \min\{D_{-n}, \frac{1}{2} d(N)\}$ then $E \leq \min\{D'_{-n}, \frac{1}{2} d'(N)\}$, so $\Delta_j = \left(\frac{1}{\mathcal{T}} - \frac{1}{\mathcal{T}'}\right) t_j E$ for each $j \neq i$, and the result follows immediately. Suppose $E \in [D_{-n}, \frac{1}{2} d(N)]$. If $i = n$ the result also holds because $D'_{-n} = D_{-n}$, $E \in [D'_{-n}, \frac{1}{2} d'(N)]$, and $\Delta_j = 0$ for $j < n$. Suppose $i < n$:

Case 1: If $E \leq D'_{-n}$ then

$$\Delta_j = \begin{cases} \left(\frac{1}{2} - \frac{E}{\mathcal{T}'}\right) d_j & \text{if } j \in N \backslash \{i, n\} \\ E - \frac{D_{-n}}{2} - \frac{E}{\mathcal{T}'} E & \text{if } j = n \end{cases}.$$

Take $j, k \in N \backslash \{i, n\}$, $j \neq k$. Now $\mathcal{T}' = D'_{-n} + E$ so $E < \frac{1}{2} \mathcal{T}'$ and $0 \leq \Delta_j \leq \Delta_k$. Moreover, $\Delta_n = \left(1 - \frac{E}{\mathcal{T}'}\right) E - \frac{1}{2} D_{-n} \geq 0$ because $E \geq D_{-n}$ and $\Delta_j \leq \Delta_n$ since $\mathcal{T}' \geq D_{-n}$.

Case 2: If $E \geq D'_{-n}$ then

$$\Delta_j = \begin{cases} 0 & \text{if } j \in N \backslash \{i, n\} \\ \frac{1}{2}(D'_{-n} - D_{-n}) & \text{if } j = n \end{cases}.$$

The result holds since $D'_{-n} \geq D_{-n}$. □

5 Progressivity and regressivity

We turn our attention to a pair of dual properties. Progressivity requires that if the claim of agent i is at most as large as the claim of agent j, agent i should receive proportionally at most as much as agent j. The dual requirement is regressivity. A rule $\mathcal{R}$ satisfies:

- *progressivity* if for each $(E, d) \in B^N$ and each pair $\{i, j\} \subset N$, if $0 < d_i \leq d_j$ then $\frac{\mathcal{R}_i(E,d)}{d_i} \leq \frac{\mathcal{R}_j(E,d)}{d_j}$.
- *regressivity* if for each $(E, d) \in B^N$ and each pair $\{i, j\} \subset N$, if $0 < d_i \leq d_j$ then $\frac{\mathcal{R}_i(E,d)}{d_i} \geq \frac{\mathcal{R}_j(E,d)}{d_j}$.

Clearly, the proportional rule is the only rule to be both regressive and progressive. The adjusted proportional rule and the minimal overlap rule satisfy neither property. But, as we show next, the adjusted proportional rule is regressive on the lower-half domain and, by duality, progressive on the higher-half domain.

Proposition 5. *The adjusted proportional rule satisfies regressivity on B_L^N and progressivity on B_H^N.*

Proof. Let $(E,d) \in B_L^N$ and denote $\mathcal{T} = \sum_{i\in N} t_i(E,d)$. Then

$$\frac{\text{APRO}_i(E,d)}{d_i} = \begin{cases} \frac{t_i E}{d_i \mathcal{T}} & \text{if } E \le \min\{D_{-n}, \frac{1}{2}d(N)\} \\ \frac{1}{2} & \text{if } D_{-n} \le E \le \frac{1}{2}d(N) \text{ and } i \in N\backslash\{n\} \\ \frac{2E-D_{-n}}{2d_n} & \text{if } D_{-n} \le E \le \frac{1}{2}d(N) \text{ and } i = n \end{cases}.$$

It follows at once that the APRO rule is regressive on B_L^N. By self-duality, one obtains that the APRO rule is progressive on B_H^N. □

Next, we focus on the minimal overlap rule. First, we present an example to exhibit that this rule is neither regressive nor progressive on the lower-half domain.

Example 1. *Let $N = \{1,2,3,4\}$, $E = 5$, and $d = (2,4,7,9)$. Clearly, $(E,d) \in B_L^N$. In fact, $d_2 < E < d_3$ so*

$$\text{MO}(E,d) = \left(\tfrac{d_1}{4}, \tfrac{d_1}{4} + \tfrac{d_2-d_1}{3}, \tfrac{d_1}{4} + \tfrac{d_2-d_1}{3} + \tfrac{E-d_2}{2}, \tfrac{d_1}{4} + \tfrac{d_2-d_1}{3} + \tfrac{E-d_2}{2}\right) = \left(\tfrac{1}{2}, \tfrac{7}{6}, \tfrac{5}{3}, \tfrac{5}{3}\right).$$

Then

$$\left(\tfrac{\text{MO}_1(E,d)}{d_1}, \tfrac{\text{MO}_2(E,d)}{d_2}, \tfrac{\text{MO}_3(E,d)}{d_3}, \tfrac{\text{MO}_4(E,d)}{d_4}\right) = \left(\tfrac{1}{4}, \tfrac{7}{24}, \tfrac{5}{21}, \tfrac{5}{27}\right).$$

Since $\frac{\text{MO}_1(E,d)}{d_1} < \frac{\text{MO}_2(E,d)}{d_2}$ the minimal overlap rule is not regressive. But, $\frac{\text{MO}_2(E,d)}{d_2} > \frac{\text{MO}_3(E,d)}{d_3}$ so the minimal overlap rule fails progressivity.

Recall that the minimal overlap rule is not self-dual. Therefore, Example 1 provides no information about whether or not the minimal overlap rule is progressive or regressive on the higher-half domain. Let us show that the minimal overlap rule satisfies a sort of progressivity with respect to the truncated claims.

Proposition 6. *Let $(E,d) \in B^N$. If $\text{MO}_{i+1}(E,d) = \text{MO}_i(E,d) + \frac{t_{i+1}-t_i}{n-i}$ for some $i \in N\backslash\{n\}$ then*

$$\frac{\text{MO}_i(E,d)}{t_i} \le \frac{\text{MO}_{i+1}(E,d)}{t_{i+1}}.$$

Proof. Let $i \in N\backslash\{n\}$ such that $\text{MO}_{i+1}(E,d) = \text{MO}_i(E,d) + \frac{t_{i+1}-t_i}{n-i}$. Simple calculations show that $\frac{\text{MO}_i(E,d)}{t_i} \le \frac{\text{MO}_{i+1}(E,d)}{t_{i+1}}$ if and only if $(t_{i+1}-t_i)\left(\frac{t_i}{n-i} - \text{MO}_i(E,d)\right) \ge 0$. But, $\left(\frac{t_i}{n-i} - \text{MO}_i(E,d)\right) \ge 0$, because:

$$(n-i)\,\text{MO}_i(E,d) = \tfrac{n-i}{n}t_1 + \cdots + \tfrac{n-i}{n-i+1}(t_i - t_{i-1}) \le t_1 + (t_2-t_1) + \cdots + (t_i - t_{i-1}) = t_i.$$

Since $(t_{i+1}-t_i) \ge 0$, we have that $(t_{i+1}-t_i)\left(\frac{t_i}{n-i} - \text{MO}_i(E,d)\right) \ge 0$. □

It turns out that the minimal overlap rule is progressive on the domain of claims problems for which no claim exceeds the endowment.

Proposition 7. *The minimal overlap rule satisfies progressivity on B_S^N.*

Proof. When $|N| = 2$ the result is clear. So, let $|N| \ge 3$ and $(E,d) \in B_S^N$. It suffices to prove that $\frac{\text{MO}_i(E,d)}{d_i} \le \frac{\text{MO}_{i+1}(E,d)}{d_{i+1}}$ for all $i \in N\backslash\{n\}$. Since $(E,d) \in B_S^N$ then $E \ge d_n$ and $t(E,d) = d$. If $E = d_n$, the result follows directly from Proposition 6. If $E > d_n$, let $s^* \in (d_{k*}, d_{k^*+1}]$, with $k^* \in \{0,1,\ldots,n-2\}$, be the solution to the equation $\sum_{i\in N} \max\{d_i - s, 0\} = E - s$. By Proposition 6, we have that $\frac{\text{MO}_i(E,d)}{d_i} \le \frac{\text{MO}_{i+1}(E,d)}{d_{i+1}}$ for all $i \in \{1,\ldots,k^*-1\}$. Now, since $s^* \ge d_{k^*}$ we have that $\text{MO}_{k^*}(E,d) = \text{MO}_{k^*}(s^*,d)$. In addition,

$$\text{MO}_{k^*+1}(E,d) = \text{MO}_{k^*+1}(s^*,d) + (d_{k^*+1} - s^*) = \text{MO}_{k^*}(s^*,d) + \frac{s^* - d_k*}{n-k^*} + (d_{k^*+1} - s^*).$$

But, again by Proposition 6,

$$\frac{\mathrm{MO}_{k^*}(E,d)}{d_{k^*}} = \frac{\mathrm{MO}_{k^*}(s^*,d)}{d_{k^*}} \leq \frac{\mathrm{MO}_{k^*+1}(s^*,d)}{d_{k^*+1}} = \frac{\mathrm{MO}_{k^*}(s^*,d) + \frac{s^*-d_{k^*}}{n-k^*}}{d_{k*+1}}$$

Then, since $\frac{d_{k*+1}-s^*}{d_{k*+1}} \geq 0$, we have that

$$\frac{\mathrm{MO}_{k^*}(E,d)}{d_{k^*}} \leq \frac{\mathrm{MO}_{k^*}(E,d) + \frac{s^*-d_{k^*}}{n-k^*}}{d_{k*+1}} + \frac{d_{k*+1} - s^*}{d_{k*+1}} = \frac{\mathrm{MO}_{k*+1}(E,d)}{d_{k*+1}}.$$

Finally, if $i \geq k^* + 1$ then $\frac{\mathrm{MO}_i(E,d)}{d_i} \leq \frac{\mathrm{MO}_{i+1}(E,d)}{d_{i+1}}$ if and only if

$$\big(\mathrm{MO}_{k^*}(s^*,d) + \tfrac{s^*-d_{k^*}}{n-k^*} + d_i - s^*\big)d_{i+1} \leq \big(\mathrm{MO}_{k^*}(E,d) + \tfrac{s^*-d_{k^*}}{n-k^*} + d_{i+1} - s^*\big)d_i,$$

Or, equivalently,

$$(d_{i+1} - d_i)\big(\mathrm{MO}_{k^*}(E,d) + \tfrac{s-d_{k^*}}{n-k^*} - s\big) \leq 0.$$

Since $d_{i+1} - d_i \geq 0$, we have to prove that

$$\mathrm{MO}_{k^*}(E,d) + \frac{s^* - d_{k^*}}{n-k^*} - s^* = \frac{(n-k^*)\,\mathrm{MO}_{k^*}(E,d) - d_{k^*} - (n-k^*-1)s^*}{n-k^*} \leq 0.$$

Indeed, the inequality holds since $n-k^*-1 \geq 0$, because $k^* \leq n-1$, and $(n-k^*)\,\mathrm{MO}_{k^*}(E,d) \leq d_{k^*}$, because $(n-k^*)\,\mathrm{MO}_{k^*}(E,d) = \frac{n-k^*}{n}d_1 + \cdots + \frac{n-k^*}{n-k^*+1}(d_{k^*} - d_{k^*-1}) \leq d_1 + d_2 - d_1 + \cdots + d_{k^*} - d_{k^*-1} = d_{k^*}$. □

The domains B_S^N and B_H^N have non-empty intersection but they are not comparable by inclusion. In any case, the minimal overlap rule also satisfies progressivity on the higher-half domain.

Proposition 8. *The minimal overlap rule satisfies progressivity on $B_H^N \cup B_{ML}^N$.*

Proof. When $|N| = 2$ the result is clear. So, let $|N| \geq 3$ and $(E,d) \in B_H^N \cup B_{ML}^N$. It suffices to prove that $\frac{\mathrm{MO}_i(E,d)}{d_i} \leq \frac{\mathrm{MO}_{i+1}(E,d)}{d_{i+1}}$ for all $i \in N\backslash\{n\}$. If $E \geq d_n$, the result follows directly from Proposition 7. As a consequence, we just have to prove the result if $D_{-n} \leq E \leq d_n$. But then $d_i \leq D_{-n} \leq E$ for all $i \in N\backslash\{n\}$, so $t(E,d) = (d_1,\ldots,d_{n-1},E)$ and $\mathrm{MO}_{i+1}(E,d) = \mathrm{MO}_i(E,d) + \frac{t_{i+1}-t_i}{n-i}$. By Proposition 6 we have that $\frac{\mathrm{MO}_i(E,d)}{d_i} \leq \frac{\mathrm{MO}_{i+1}(E,d)}{d_{i+1}}$ for each $i \in N\backslash\{n, n-1\}$. It remains to be proved that $\mathrm{MO}_n(E,d)d_{n-1} \geq \mathrm{MO}_{n-1}(E,d)E$. Now, this inequality holds if and only if $(E - d_{n-1})(d_{n-1} - \mathrm{MO}_{n-1}(E,d)) \geq 0$. But,

$$\mathrm{MO}_{n-1}(E,d) = \tfrac{d_1}{n} + \tfrac{d_2-d_1}{n-1} + \cdots + \tfrac{d_{n-1}-d_{n-2}}{2} \leq \tfrac{d_1}{2} + \tfrac{d_2-d_1}{2} + \cdots + \tfrac{d_{n-1}-d_{n-2}}{2} = \tfrac{d_{n-1}}{2}.$$

Since $E \geq d_{n-1}$, we conclude that, indeed, $(E - d_{n-1})(d_{n-1} - \mathrm{MO}_{n-1}(E,d)) \geq 0$. □

6 Variable-population properties

Consider now situations in which the population of claimants involved may vary. In this case, a claims problem is defined by first specifying $N \in \mathcal{N}$, then a pair $(E,d) \in B^N$. So, a rule is a function defined on $\bigcup_{N\in\mathcal{N}} B^N$ that associates with each $N \in \mathcal{N}$ and each $(E,d) \in B^N$ an awards vector for (E,d). We say that a rule $\mathcal{R}$ satisfies:

- *null claims consistency*, if for each $N \in \mathcal{N}$, each $(E,d) \in B^N$ and each $N' \subset N$, if $d_i = 0$ for all $i \in N\backslash N'$, then $\mathcal{R}_{N'}(E,d) = \mathcal{R}(E, d_{N'})$.
- *population monotonicity*, if for each $N \in \mathcal{N}$, each $(E,d) \in B^N$, and each $N' \subset N$, if $d(N') \geq E$ then $\mathcal{R}_{N'}(E,d) \leq \mathcal{R}(E, d_{N'})$.
- *linked endowment-population monotonicity*, if for each $N \in \mathcal{N}$, each $(E,d) \in B^N$, and each $N' \subset N$, if $d(N') \geq E$ then $\mathcal{R}_{N'}(E,d) \leq \mathcal{R}(E + d(N\backslash N'), d)$.

- *order preservation under population variations*, if for each $(E,d) \in B^N$, each $i \in N$ with $E < d(N\backslash\{i\})$ and each pair $\{j,k\} \subseteq N\backslash\{i\}$, if $d_j \leq d_k$, then $\mathcal{R}_k(E,d) - \mathcal{R}_k(E,d_{-i}) \leq \mathcal{R}_j(E,d) - \mathcal{R}_j(E,d_{-i})$.
- *order preservation under the reduction operation*, if for each $(E,d) \in B^N$, each $i \in N$ with $E > d_i$ and each pair $\{j,k\} \subseteq N\backslash\{i\}$, if $d_j \leq d_k$, then $\mathcal{R}_j(E,d) - \mathcal{R}_j(E-d_i,d_{-i}) \leq \mathcal{R}_k(E,d) - \mathcal{R}_k(E-d_i,d_{-i})$.

Null claims consistency is a weak requirement and it is satisfied by the ten rules. Population monotonicity and linked endowment-population monotonicity are dual properties. The dual property of order preservation under population variations is order preservation under the reduction operation. As Thomson (2019) points out, null claims consistency and other-regarding claim monotonicity together imply population monotonicity. Similarly, null claims consistency and order preservation under claims variation together imply order preservation under population variation.

Proposition 9. *Let $\mathcal{R}$ be a rule satisfying null claims consistency.*

1. *If $\mathcal{R}$ satisfies other-regarding claim monotonicity then $\mathcal{R}$ satisfies population monotonicity.*
2. *If $\mathcal{R}$ satisfies order preservation under claims variation then $\mathcal{R}$ satisfies order preservation under population variation.*

Proof. We just prove the second statement. Let $N \in \mathcal{N}$, $(E,d) \in B^N$, $i \in N$ with $E < d(N\backslash\{i\})$, and $\{j,k\} \subseteq N\backslash\{i\}$ such that $d_j \leq d_k$. Then,

$$\mathcal{R}_k(E,d) - \mathcal{R}_k(E,d_{-i}) = \mathcal{R}_k(E,d) - \mathcal{R}_k(E,(d_{-i},0)) \leq \mathcal{R}_j(E,d) - \mathcal{R}_j(E,(d_{-i},0)) = \mathcal{R}_j(E,d) - \mathcal{R}_j(E,d_{-i}),$$

where the inequality holds by order preservation under claim variation, and the equalities by null claims consistency. Therefore $\mathcal{R}$ satisfies order preservation under population variation. □

It is known, see Thomson (2019) and Bosmans and Lauwers (2011), that the minimal overlap rule satisfies other-regarding claim monotonicity, population monotonicity, and order preservation under claim variation. Therefore, by Proposition 9, it also satisfies order preservation under population variation.

The following example, adapted from Grahn and Voorneveld (2002),[2] illustrates that the adjusted proportional rule does not satisfy neither population monotonicity nor linked endowment-population monotonicity.

Example 2. *Let $N = \{1,2,3,4\}$, $E = 12$, and $d = (1,2,9,10)$. Consider the problems $(E,d) \in B^N$ and $(E,d_{-2}) \in B^{N\backslash\{2\}}$. Then*

$$\text{APRO}(E,d) = \left(\tfrac{6}{11}, \tfrac{12}{11}, \tfrac{54}{11}, \tfrac{60}{11}\right),\ \text{APRO}(E,d_{-2}) = \left(\tfrac{9}{17}, \tfrac{89}{17}, \tfrac{106}{17}\right).$$

So, when claimant 2 leaves, claimant 1 receives less than initially, and hence the adjusted proportional rule is not population monotonic. Obviously, since the APRO *rule is self-dual, it also violates linked endowment-population monotonicity.*

According to Thomson (2019) and Mirás Calvo et al. (2020), the other nine rules that we have considered satisfy population monotonicity and linked endowment-population monotonicity. Recall from Proposition 3 that the adjusted proportional rule does not satisfy other-regarding claim monotonicity. We reach the same conclusion from Proposition 9, because the adjusted proportional rule satisfies null claims consistency but violates population monotonicity.

We conclude the Section studying on which half domain the adjusted proportional rule satisfies the monotonicity properties listed above.

Proposition 10. *The adjusted proportional rule satisfies population monotonicity on the lower-half domain and linked endowment-population monotonicity on the higher-half domain.*

Proof. Let $N \in \mathcal{N}$, $(E,d) \in B_L^N$, and $N' \subset N$ such that $d(N') \geq E$. If $(E,d_{N'}) \in B_L^{N'}$, then by Proposition 4, $\text{APRO}_{N'}(E,d) \leq \text{APRO}_{N'}(E,(d_{N'},0_{N\backslash N'}))$. But, the adjusted proportional rules satisfies null claims consistency, so $\text{APRO}_{N'}(E,(d_{N'},0_{N\backslash N'})) = \text{APRO}_{N'}(E,d_{N'})$. Therefore, $\text{APRO}_{N'}(E,d) \leq \text{APRO}_{N'}(E,d_{N'})$. On the contrary, if $(E,d_{N'}) \in B_H^{N'}$ then, for each $j \in N'$,

$$\text{APRO}_j(E,d) \leq \text{APRO}_j(\tfrac{1}{2}d(N),d) = \tfrac{d_j}{2} = \text{APRO}_j(\tfrac{1}{2}d(N'),d_{N'}) \leq \text{APRO}_j(E,d_{N'})$$

[2]In fact, we present the dual version of the example in Grahn and Voorneveld (2002), since these authors show that the adjusted proportional rule does not satisfy linked endowment-population monotonicity, that they call the thieve property.

where, we have applied that $E \leq \frac{1}{2}d(N)$ and that the adjusted proportional rule satisfies endowment monotonicity and the midpoint property. Hence, we conclude that the adjusted proportional rule satisfies population monotonicity on B_L^N. Since population monotonicity and linked endowment-population monotonicity are dual properties and the adjusted proportional rule is self-dual, the second statement follows immediately. □

Finally, we see that the adjusted proportional rule satisfies order preservation under population variation on the lower-half domain and its dual property on the corresponding dual domain.

Proposition 11. *The adjusted proportional rule satisfies order preservation under population variation on the lower-half domain and order preservation under the reduction problem on the higher-half domain.*

Proof. Let $(E,d) \in B_L^N$, $i \in N$ with $E < d(N \backslash \{i\})$ and a pair $\{j,k\} \subseteq N \backslash \{i\}$ where $d_j \leq d_k$, we have to prove that $\text{APRO}_k(E,d) - \text{APRO}_j(E,d) \leq \text{APRO}_k(E,d_{-i}) - \text{APRO}_j(E,d_{-i})$. When $E \leq \frac{1}{2}D_{-i}$, the property holds directly by Proposition 4 since the APRO rule satisfies null claims consistency. Now, if $E \geq \frac{1}{2}D_{-i}$, we have that

$$\begin{aligned}\text{APRO}_k(E,d_{-i}) - \text{APRO}_j(E,d_{-i}) &\geq \text{APRO}_j(E,d_{-i})\tfrac{d_k}{d_j} - \text{APRO}_j(E,d_{-i}) = \left(\tfrac{d_k-d_j}{d_j}\right)\text{APRO}_j(E,d_{-i})\\ &\geq \left(\tfrac{d_k-d_j}{d_j}\right)\text{APRO}_j(E,d) = \text{APRO}_j(E,d)\tfrac{d_k}{d_j} - \text{APRO}_j(E,d)\\ &\geq \text{APRO}_k(E,d) - \text{APRO}_j(E,d).\end{aligned}$$

The first inequality holds because, by Proposition 5, the APRO rule is progressive on $B_H^{N\backslash\{i\}}$. The second inequality is an application of Proposition 10, and the last one of Proposition 5. Since order preservation under population variation and order preservation under the reduction problem are dual properties and the adjusted proportional rule is self-dual, the second part follows immediately. □

7 Lorenz-based characterizations

Given a claims problem $(E,d) \in B^N$, an awards vector $x \in X(E,d)$ Lorenz-dominates an awards vector $y \in X(E,d)$ if all the cumulative sums of the rearranged coordinates are greater with x than with y.

Definition 1. *Let $x,y \in \mathbb{R}^n_{\leq}$. We say that x Lorenz-dominates y, and write $x \succeq y$, if for each $k = 1,\ldots,n-1$*

$$\sum_{j=1}^{k} x_j \geq \sum_{j=1}^{k} y_j \text{ and } \sum_{j=1}^{n} x_j = \sum_{j=1}^{n} y_j.$$

The Lorenz order is a partial order in $\mathbb{R}^n_{\leq}$, so it is a binary relation that is reflexive, antisymmetric, and transitive. If x Lorenz-dominates y and $x \neq y$, then at least one of the $n-1$ inequalities is strict.

We have assumed that given a claims problem $(E,d) \in B^N$ the claims vector has its coordinates ordered from small to large, that is, $d \in \mathbb{R}^n_{\leq}$. Moreover, the ten rules satisfy order preservation in awards. So if $\mathcal{R}$ is any of these rules then $\mathcal{R}(E,d) \in \mathbb{R}^n_{\leq}$. Therefore, we can use the Lorenz-dominance criterion to check whether a rule is more favorable to smaller claimants relative to larger claimants than other. This analysis allows to classify, and therefore, recommend specific rules depending on the different contexts of application: taxation, bankruptcy, pensions,...

Definition 2. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rules that satisfy order preservation in awards. We say that $\mathcal{R}$ Lorenz-dominates $\mathcal{R}'$ on the subdomain $\Omega \subset B^N$, and we write $\mathcal{R} \succeq \mathcal{R}'$, if $\mathcal{R}(E,d) \succeq \mathcal{R}'(E,d)$ for all $(E,d) \in \Omega$.*

The proportional rule can be compared using the Lorenz criterion with the rules that are progressive or regressive. Given a subdomain $\Omega \subset B^N$, the proportional rule Lorenz-dominates on Ω any rule that is progressive on Ω. Naturally, the proportional rule is Lorenz-dominated on Ω by any rule that is regressive on Ω.

Proposition 12. *Let $\Omega \subset B^N$ and let $\mathcal{R}$ be a rule that satisfies order preservation in awards.*

1. *If $\mathcal{R}$ is progressive on $\Omega \subset B^N$ then the proportional rule Lorenz-dominates $\mathcal{R}$ on Ω.*

2. *If $\mathcal{R}$ is regressive on $\Omega \subset B^N$ then the proportional rule is Lorenz-dominated by $\mathcal{R}$ on Ω.*

Proof. Let $\mathcal{R}$ be a rule that satisfies order preservation in awards and that is progressive on $\Omega \subset B^N$. Given $(E,d) \in \Omega$ and $k \in N \backslash \{n\}$ we have to prove that $\sum_{j=1}^{k} \mathcal{R}_j(E,d) \leq \sum_{j=1}^{k} \text{PRO}_j(E,d) = \frac{E}{d(N)} \sum_{j=1}^{k} d_j$ or, equivalently, $d(N) \sum_{j=1}^{k} \mathcal{R}_j(E,d) \leq E \sum_{j=1}^{k} d_j$. Since $E = \sum_{j=1}^{n} \mathcal{R}_j(E,d)$, the last inequality can be written as

$$\Big(\sum_{j=1}^{k} d_j + \sum_{i=k+1}^{n} d_i\Big) \sum_{j=1}^{k} \mathcal{R}_j(E,d) \leq \Big(\sum_{j=1}^{k} R_j(E,d) + \sum_{i=k+1}^{n} R_i(E,d)\Big) \sum_{j=1}^{k} d_j.$$

Therefore, we have to show that $\sum_{j=1}^{k} \mathcal{R}_j(E,d) \sum_{i=k+1}^{n} d_i \leq \sum_{i=k+1}^{n} \mathcal{R}_i(E,d) \sum_{j=1}^{k} d_j$. If $\mathcal{R}$ is progressive on $\Omega \subset B^N$ then $d_i R_j(E,d) \leq R_i(E,d) d_j$ for all $j \in \{1, \dots k\}$ and all $i \in \{k+1, \dots n\}$. Therefore, $\mathcal{R}_j(E,t) \sum_{i=k+1}^{n} d_i \leq d_j \sum_{i=k+1}^{n} \mathcal{R}_i(E,d)$ for all $j \in \{1, \dots, k\}$. But then $\sum_{j=1}^{k} \mathcal{R}_j(E,d) \sum_{i=k+1}^{n} d_i \leq \sum_{i=k+1}^{n} \mathcal{R}_i(E,d) \sum_{j=1}^{k} d_j$ and, indeed, $\text{PRO}(E,d) \succeq \mathcal{R}(E,d)$.

On the other hand, if $\mathcal{R}$ is regressive on $\Omega \subset B^N$, a similar argument shows that $\mathcal{R}(E,d) \succeq \text{PRO}(E,d)$, which proves the second statement. □

If two rules $\mathcal{R}$ and $\mathcal{R}'$ satisfy self-duality, then $\mathcal{R}$ and $\mathcal{R}'$ cannot be compared on the entire domain B^N. It is well known that the claims truncation operator preserves the Lorenz order. The same happens with the attribution of minimal rights. Besides, the duality operator reverses the Lorenz order. We formally state these results.

Proposition 13. *Let $\mathcal{R}$ and $\mathcal{R}'$ be two rules satisfying order preservation in awards and $\Omega \in B^N$ a domain of claims problems.*

1. *If $\mathcal{R} \succeq \mathcal{R}'$ then $\mathcal{R}(E, t(E,d)) \succeq \mathcal{R}'(E, t(E,d))$ for all $(E,d) \in B^N$.*
2. *If $\mathcal{R} \succeq \mathcal{R}'$ then $\mathcal{R}^m \succeq (\mathcal{R}')^m$ where $\mathcal{R}^m(E,d) = m(E,d) + \mathcal{R}\big(E - \sum_{i \in N} m_i(E,d), d - m(E,d)\big)$ for all $(E,d) \in B^N$.*
3. *If $\mathcal{R}$ and $\mathcal{R}'$ satisfy order preservation in losses then $\mathcal{R} \succeq \mathcal{R}'$ on Ω if and only if $(\mathcal{R}')^* \succeq \mathcal{R}^*$ on Ω^*.*

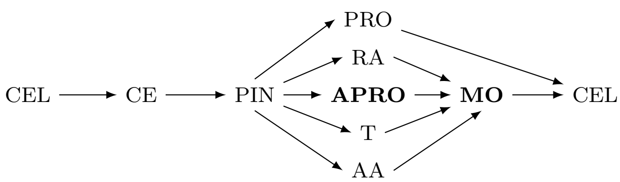

Figure 4: Ranking of rules on the domain B^N.

We reproduce in Figure 4 a diagram that summarizes the ranking of the ten rules on B^N using the Lorenz criterion: an arrow (or a sequence of arrows) from a rule $\mathcal{R}$ to a rule $\mathcal{R}'$ indicates that $\mathcal{R}$ Lorenz-dominates $\mathcal{R}'$, and the absence of an arrow (or of a sequence of arrows) indicates that there is no relationship. In particular, Corollary 13.3 in Thomson (2019) gives a direct prove of the arrow T → MO.

Proposition 14 (Thomson (2019))**.** *The Talmud rule Lorenz-dominates the minimal overlap rule.*

The arrows involving the average of awards rule, PIN → AA → MO, are due to Mirás Calvo et al. (2021). Bosmans and Lauwers (2011) provide Lorenz-based characterizations of the constrained equal awards, constrained equal losses, Talmud, Piniles', constrained egalitarian, and minimal overlap rules, from which they deduce the other arrows in Figure 4. Let us state the corresponding characterization of the minimal overlap rule.

Proposition 15 (Bosmans and Lauwers (2011))**.** *Let $\mathcal{S}$ be the set of rules that satisfy order preservation in awards, order preservation in losses, order preservation under claims variations, null claims consistency, and $\frac{1}{|N|}$-truncated-claims lower bounds on awards. The minimal overlap rule is the only rule in $\mathcal{S}$ that is Lorenz-dominated by each rule in $\mathcal{S}$.*

Naturally, from Proposition 15, we know that the minimal overlap rule satisfies order preservation under claims variations. Now, in order to prove that the adjusted proportional rule Lorenz-dominates the minimal overlap rule, Bosmans and Lauwers (2011) rely on Proposition 15. But, according to Proposition 3, the adjusted proportional rule violates order preservation under claims variations, so that comparison cannot be inferred from the given characterization of the minimal overlap rule. Therefore, arrow APRO $\rightarrow$ MO on Figure 4 has not being properly established in Bosmans and Lauwers (2011).

Our goal is to prove that, indeed, the adjusted proportional rule Lorenz-dominates the minimal overlap rule. First, we show that the awards vector chosen for each problem by the adjusted proportional rule is minimal on the lower-half domain among the awards chosen for that problem by any rule satisfying the midpoint property, minimal rights first, claims truncation invariance, order preservation, and regressivity on B_L^N.

Theorem 1. *Let $\mathcal{S}_1$ be the set of rules satisfying the midpoint property, minimal rights first, claims truncation invariance, order preservation in awards, and regressivity on B_L^N. Let $\mathcal{S}_2$ be the set of rules satisfying the midpoint property, minimal rights first, claims truncation invariance, order preservation in losses, and progressivity on B_H^N.*

1. *The adjusted proportional rule is the only rule in $\mathcal{S}_1$ that is Lorenz-dominated by each rule in $\mathcal{S}_1$ on the lower-half domain.*

2. *The adjusted proportional rule is the only rule in $\mathcal{S}_2$ that Lorenz-dominates each rule in $\mathcal{S}_2$ on the higher-half domain.*

Proof. Restrict to the domain B_L^N. First, let us show that the APRO rule is the only rule in $\mathcal{S}_1$ that is Lorenz-dominated by each rule in $\mathcal{S}_1$ on B_L^N. Clearly, APRO $\in \mathcal{S}_1$. Let $(E,d) \in B_L^N$ and $\mathcal{R} \in \mathcal{S}_1$. By claims truncation invariance, $\mathcal{R}(E,d) = \mathcal{R}(E,t)$. If $E \leq D_{-n}$ then $\text{APRO}(E,d) = \text{PRO}(E,t)$. But, $\mathcal{R}_1(E,t) \geq \text{PRO}_1(E,t)$ if and only if $\mathcal{R}_1(E,t)\sum_{j=2}^{n} t_j \geq t_1 \sum_{j=2}^{n} \mathcal{R}_j(E,t)$. This inequality holds because $\mathcal{R}$ is regressive. Let $\mathcal{T} = \sum_{j\in N} t_j$. Now, for each $k \in N\backslash\{n\}$, $\sum_{j=1}^{k} \mathcal{R}_j(E,t) \geq \sum_{j=1}^{k} \text{PRO}_j(E,t) = \frac{E}{\mathcal{T}} \sum_{j=1}^{k} t_j$ if and only if $\sum_{j=1}^{k} \mathcal{R}_j(E,t) \sum_{j=k+1}^{n} t_j \geq \sum_{j=k+1}^{n} \mathcal{R}_j(E,t) \sum_{j=1}^{k} t_j$. Again, by regressivity, $\mathcal{R}_j(E,t) \sum_{j=k+1}^{n} t_j \geq t_j \sum_{j=k+1}^{n} \mathcal{R}_j(E,t)$ for $j \in \{1,\ldots,k\}$. If $E \in [D_{-n}, \frac{1}{2}d(N)]$ then $m = (0,\ldots,0,E-D_{-n})$. By the midpoint property, $\mathcal{R}(E,t) = m + \mathcal{R}(D_{-n},(d_{-n},D_{-n})) = m + \text{APRO}(D_{-n},(d_{-n},D_{-n})) = \text{APRO}(E,t)$.

Restrict to the domain B_H^N. We claim that the APRO rule is the only rule in $\mathcal{S}_2$ that Lorenz-dominates each rule in $\mathcal{S}_2$ on B_H^N. Indeed, APRO $\in \mathcal{S}_2$. But, order preservation in awards and order preservation in losses are dual properties, and the same happens with claims truncation invariance and minimal rights first, and with regressivity and progressivity. Besides, B_L^N and B_H^N are dual domains. Therefore, the characterization on B_H^N is the dual theorem of the one just proven above. □

Now, in order to prove that the adjusted proportional rule Lorenz-dominates the minimal overlap rule, one has to show that the adjusted proportional rule Lorenz-dominates the minimal overlap rule on both half domains.

Theorem 2. *The adjusted proportional rule Lorenz-dominates the minimal overlap rule.*

Proof. Let us show first that APRO $\succeq$ MO on B_L^N. The adjusted proportional rule and the minimal overlap rule both satisfy claims truncation invariance, therefore given $(E,d) \in B_L^N$ we have that $\text{APRO}(E,d) = \text{APRO}(E,t)$ and $\text{MO}(E,d) = \text{MO}(E,t)$. Note that $(E,t) \in B_S^N$ because $E \geq t_n(E,d) = \min\{E, d_n\}$. Therefore, by Proposition 8 and Proposition 12, we have that $\text{PRO}(E,t) \succeq \text{MO}(E,t)$.

Now, if $E \leq \min\{D_{-n}, \frac{1}{2}d(N)\}$ then, by Proposition 2, $\text{APRO}(E,d) = \text{PRO}(E,t)$. Hence, $\text{APRO}(E,d) = \text{PRO}(E,t) \succeq \text{MO}(E,t) = \text{MO}(E,d)$. On the other hand, if $E \in [D_{-n}, \frac{1}{2}d(N)]$ then, by Corollary 1, $\text{APRO}(E,d) = \text{T}(E,d)$. Applying Proposition 14 we conclude that $\text{APRO}(E,d) = \text{T}(E,d) \succeq \text{MO}(E,d)$.

According to Theorem 1 we have that APRO $\succeq$ T on B_H^N. On the other hand, by Proposition 14, T $\succeq$ MO. Therefore, APRO $\succeq$ MO on B_H^N. □

Let us present an example of a three-claimant problem that illustrates the Lorenz-dominance results relating the proportional, adjusted proportional, and minimal overlap rules on the lower-half, higher-half, and middle domains.

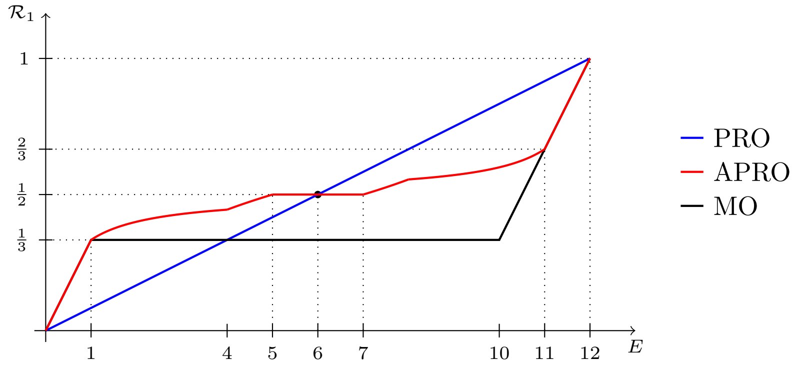

Figure 5: First coordinate of the PRO, APRO, and MO rules as a function of E when $d = (1, 4, 7)$.

Example 3. *Let* $N = \{1, 2, 3\}$ *and* $d = (1, 4, 7) \in \mathbb{R}^3_{\leq}$. *Then* $d(N) = 12$, $\frac{1}{2}d(N) = 6$, *and* $D_{-3} = d_1 + d_2 = 5$, *so* $D_{-3} < \frac{1}{2}d(N) < d_3$. *If* $E \in [5, 7]$ *then* $(E, d) \in B^N_M$ *so* $\mathrm{APRO}(E, d) = (\frac{1}{2}, 2, E - \frac{19}{2})$. *Now, let* $\mathcal{R}$ *be any of these three rules:* PRO, APRO, *or* MO. *Consider the function* $\mathcal{R}_1(\cdot, d)\colon [0, 12] \to \mathbb{R}$ *that assigns to each* $E \in [0, 12]$ *the value* $\mathcal{R}_1(E, d)$, *the award given to the first claimant by* $\mathcal{R}$ *in the claims problem* (E, d). *Figure 5 shows the graphs corresponding to the three rules. Observe that,* $\mathrm{APRO}_1(E, d) \geq \mathrm{MO}_1(E, d)$ *for all* $E \in [0, 12]$. *Moreover,* $\mathrm{APRO}_1(E, d) \geq \mathrm{PRO}_1(E, d)$ *if* $E \in [0, 6]$ *and* $\mathrm{APRO}_1(E, d) \leq \mathrm{PRO}_1(E, d)$ *if* $E \in [6, 12]$. *Also,* $\mathrm{PRO}_1(E, d) \geq \mathrm{MO}_1(E, d)$ *if* $E \in [5, 12]$. *Nevertheless, the paths of* $\mathrm{PRO}_1(\cdot, d)$ *and* $\mathrm{MO}_1(\cdot, d)$ *cross each other on the interval* $[0, 5]$.

8 Summary

The importance of the midpoint, the lower-half, the higher-half, and the simple claims domains has already being remarked in the literature on claims problems. Also of interest is the middle domain, because the set of awards vectors of any claims problem that belongs to this domain has a very simple structure. Proposition 1 and Corollary 1 show that the Talmud, the random arrival, and the average of awards rules coincide when restricted to the middle domain.

In the first part of the paper, sections 4 to 6, we have analyzed whether or not the adjusted proportional and the minimal overlap rules satisfy a number of properties on the above mentioned domains. Table 2 summarizes our findings, with the reference to the Proposition or the Example where the corresponding mark is justified (the ones without reference were already known in the literature).

	APRO		MO	
	B^N_L	B^N_H	B^N_L	B^N_H
Other-regarding claim monotonicity	✓ (P4)	– (P3)	✓	✓
Order preservation under claim variation	✓ (P4)	– (P3)	✓	✓
Progressivity	✓ (P5)	–	– (E1)	✓ (P8)
Regressivity	–	✓ (P5)	– (E1)	–
Population monotonicity	✓ (P10)	– (E2)	✓	✓
Order preservation under population variation	✓ (P11)	–	✓ (P9)	✓ (P9)

Table 2: Properties satisfied by the APRO and MO rules on the lower-half and higher-half domains.

Section 7 is devoted to compare by means of the Lorenz criterion the adjusted proportional and the minimal overlap rule. We see in Proposition 12 that, if restricted to a given domain, the proportional rule Lorenz-dominates the rules that are progressive on this domain and is Lorenz-dominated by the rules that are regressive on this domain. Theorem 1 gives two characterizations of the adjusted proportional rule on both the lower and higher half domains. Using these characterizations, Mirás Calvo et al. (2021) show that the average of awards rule Lorenz-dominates the adjusted proportional rule on the lower-half domain and it is Lorenz-dominated by the adjusted proportional rule on the higher-half domain. We prove in Theorem 2 that the adjusted proportional rule Lorenz-dominates the minimal overlap rule.

Figure 4 shows the ranking of the ten rules in the domain of claims problems. Now, we can produce similar diagrams on the lower-half, the middle, the midpoint, and the higher-half domains. Figure 6 summarizes the ranking of the ten rules on these domains. The arrows have been compiled from Bosmans and Lauwers (2011), Thomson (2019), Mirás Calvo et al. (2021), and our results.

Acknowledgments

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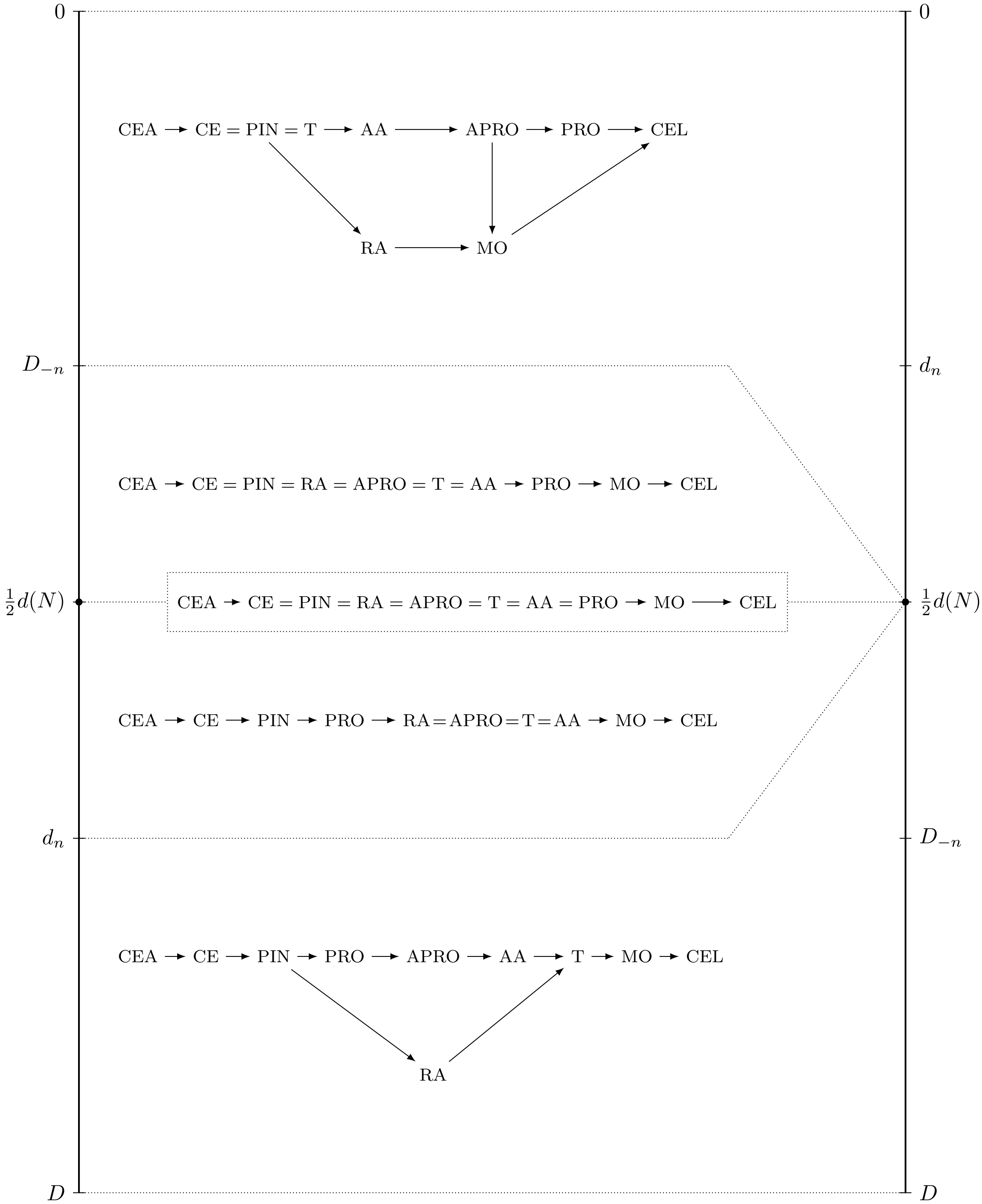

Figure 6: Ranking of rules on the lower-half, middle, midpoint, and higher-half domains.