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The core-center rule for the bankruptcy problem

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Abstract

Given a bankruptcy problem, the core-center rule selects the mean value of all the award vectors bounded from below by the minimal rights and from above by the truncated claims, that is, the center of gravity of the core of the associated bankruptcy game. We show that this rule satisfies a good number of properties so as to be included in the inventory of solutions for this class of problems, among them, homogeneity, continuity, self-duality, claim monotonicity, and order preservation. As in other contexts where the center of gravity plays an important role in studying certain properties of a system, the core-center rule provides a great insight on the behavior of the core of a bankruptcy game. In particular, we analyze in detail how the core changes when the initial endowment increases, and prove several additional properties that the core-center rule satisfies: endowment monotonicity, population monotonicity, other-regarding claim monotonicity and $\frac{1}{|N|}$-truncated-claims lower bounds on awards.

Keywords: bankruptcy problems, bankruptcy games, core, core-center rule

1 Introduction

A bankruptcy problem arises when a scarce resource, or initial endowment, has to be shared among a set of claimants, with the same rights, that claim different amounts of it and with the condition that the endowment is not sufficient to fully satisfying all the requests. The question is, how to select a division between the claimants of the amount available. Formally, a rule is a function that associates with each bankruptcy problem an awards vector whose coordinates add up to the endowment and such that no claimant should be asked to pay and no claimant should be awarded more than her claim. There are several rules that are commonly used in practice or discussed in theoretical work, for instance, the proportional rule, the constrained equal awards rule, the constrained equal losses rule, the Talmud rule, and the random arrival rule. Some of these rules have a direct definition that has some sort of an intuitive appeal. Other rules are selected according to the properties that they satisfy or violate. Thomson (2019) provides a comprehensive survey of the rules and the relevant properties.

Another way of defining meaningful rules is by associating to each claims problem a cooperative game and using game theory to come out with a solution. To each bankruptcy problem one can associate a coalitional game, with the claimants as players, and the corresponding characteristic function that assigns to any coalition what is left from the initial endowment, if any, once the claims of the members of the complementary coalition have been met. The division rules that correspond to a solution to coalitional games belong to the core of the associated bankruptcy game. The core of a bankruptcy game is a non-empty convex polytope with a particular structure: it is the intersection of a hyperrectangle with the efficiency hyperplane (Curiel et al., 1987).

The center of gravity has proved very useful in the study of many systems. The core-center, the center of gravity (centroid) of the core, was introduced and characterized for the general class of balanced games

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by González-Díaz and Sánchez-Rodríguez (2007, 2009) and was studied on the domain of airport games by González-Díaz et al. (2015, 2016) and Mirás Calvo et al. (2016). The core-center rule assigns to each bankruptcy problem the mathematical expectation of the uniform distribution over the core of the associated bankruptcy game. Our goal is to show that it satisfies a good number of significant properties and that, consequently, it may be an interesting addition to the inventory of division rules. From the general properties of the core-center of an arbitrary balanced game we show that the core-center rule satisfies minimal rights first, claims truncation invariance, equal treatment of equals, anonymity, homogeneity, and continuity. Analyzing the core-center rule demands a detailed examination of the structure of the core of the associated bankruptcy game. We prove that the core of a bankruptcy game is the translate by the vector of claims of the minus core of the bankruptcy game associated with the dual bankruptcy problem. Therefore, the core-center rule is self-dual. We also provide two different partitions of the core of a bankruptcy game as the union of two pieces with negligible intersection that are, in turn, cores of bankruptcy games. These decompositions provide geometrical interpretations of order preservation and claim monotonicity respectively, that allow us to show that the core-center rule satisfies both properties.

One essential aspect in our analysis of the core of a bankruptcy game is to describe how it changes when the endowment increases. When there are at least three claimants, the volume of the core of a bankruptcy game, as a function of the endowment, is a differentiable function. Therefore, we can use differential calculus to express some properties of the core-center rule in terms of the derivatives of the volume function. As a result, we rely on an integral representation of the core-center rule to show that it is a differentiable function of the endowment and that it satisfies endowment monotonicity, population monotonicity, other-regarding claim monotonicity, and $\frac{1}{|N|}$-truncated-claims lower bounds on awards. The core-center rule coincides with the concede-and divide rule for two claimants but, for larger populations, it differs from the standard rules. As a direct consequence of well known axiomatic characterizations of some of these rules we conclude that the core-center rule violates composition up, composition down, and consistency. The integral formula for the core-center rule admits an interpretation in terms of the reduced games introduced by Davis and Maschler (1965). If a claimant leaves, the core-center rule selects, for each of the remaining claimants, the weighted average of the choices made by the rule over all of the reduced problems where the claimant who leaves receives an award between her minimal right and truncated claim. This property, that we call single-agent weighted consistency, characterizes the core-center rule.

In Section 2 we introduce the basic definitions and notations. Section 3 explores the special structure of the core of a bankruptcy game. We relate the core of a bankruptcy game and the cores of the reduced games. We define the core-center rule and prove its basic properties in Section 4. We show that the core-center rule satisfies endowment differentiability in Section 5 and provide an expression for it as the weighted average of the choices made by the rule in each of the reduced problems. In Section 6 we see that the core-center rule satisfies endowment monotonicity and, as a consequence, other monotonicity and lower bounds properties. Finally, in Section 7 we discuss some issues concerning the computation of the core-center rule, its relation to other bankruptcy rules, and an axiomatic characterization of this rule. We include an Appendix with the computations and results that are just technical in nature. The results in the Appendix are grouped in five parts: (A) the volume of the core of a bankruptcy game, (B) the computation of the core-center rule, (C) decompositions of the core of a bankruptcy game, (D) integral representations of the core-center rule, and (E) bounds for the core-center rule.

2 Preliminaries

Let $\mathcal{N}$ be the set of all finite subsets of the natural numbers $\mathbb{N}$. A bankruptcy problem (O'Neill, 1982; Aumann and Maschler, 1985) with set of claimants $N \in \mathcal{N}$ is a pair (E, d) where $E \geq 0$ is the endowment to be divided and $d \in \mathbb{R}^N$ is the vector of claims satisfying $d_i \geq 0$ for all $i \in N$ and $\sum_{i \in N} d_i \geq E$. We denote the class of bankruptcy problems with set of players N by B^N. A bankruptcy rule is a function $\mathcal{R} : B^N \to \mathbb{R}^N$ assigning to each bankruptcy problem $(E, d) \in B^N$ an awards vector $\mathcal{R}(E, d) \in \mathbb{R}^N$ such that $0 \leq \mathcal{R}_i(E, d) \leq d_i$ for every $i \in N$ and $\sum_{i \in N} \mathcal{R}_i(E, d) = E$, that is, a way of associating with each bankruptcy problem a division of the amount available between the claimants. The minimal right of claimant $i \in N$ in $(E, d) \in B^N$ is the quantity $m_i(E, d) = \max\big\{0, E - \sum_{k \neq i} d_k\big\}$, what is left of the endowment after all other claimants have been fully compensated if possible, and 0 otherwise. The truncated claim of claimant $i \in N$ in $(E, d) \in B^N$ is $t_i(E, d) = \min\{E, d_i\}$, the minimun of the claim and the endowment. Let $m(E, d) = \big(m_i(E, d)\big)_{i \in N}$ and

$t(E,d) = \big(t_i(E,d)\big)_{i\in N}$.

Given $x \in \mathbb{R}^N$ and $N' \subset N$ let $x_{N'} = \big(x_i\big)_{i\in N'} \in \mathbb{R}^{N'}$ be the projection of x onto $\mathbb{R}^{N'}$. In particular denote $x_{-i} = x_{N\setminus\{i\}} \in \mathbb{R}^{N\setminus\{i\}}$ the vector obtained by neglecting the ith-coordinate of x, i.e., $x_{-i} = (x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_n)$. For simplicity, we will write $x = (x_{-i}, x_i)$. Also, let $x(N') = \sum_{i\in N'} x_i$. Following Thomson (2019), we present some standard properties of rules. We say that a rule $\mathcal{R}$ satisfies:

- *minimal rights first*, if for each $(E,d) \in B^N$ we have $\mathcal{R}(E,d) = m(E,d) + \mathcal{R}\big(E - \sum_{i\in N} m_i(E,d), d - m(E,d)\big)$.
- *claims truncation invariance*, if for each $(E,d) \in B^N$ we have $\mathcal{R}(E,d) = \mathcal{R}(E, t(E,d))$.
- $\frac{1}{|N|}$*-truncated-claims lower bounds on awards*, if for each $(E,d) \in B^N$ we have $\mathcal{R}(E,d) \geq \frac{1}{|N|} t(E,d)$.
- $\frac{1}{|N|}$*-min-of-claim-and-deficit lower bounds on losses*, if for each $(E,d) \in B^N$ we have $d - \mathcal{R}(E,d) \geq \frac{1}{|N|} t\big(d(N) - E, d\big)$.
- *min-of-claim-and-equal-division lower bounds on awards*, if for each $(E,d) \in B^N$ we have $\mathcal{R}(E,d) \geq t\big(\frac{E}{|N|}, d\big)$.
- *equal treatment of equals*, if for each $(E,d) \in B^N$ and each $\{i,j\} \subset N$, if $d_i = d_j$ we have $\mathcal{R}_i(E,d) = \mathcal{R}_j(E,d)$.
- *anonymity*, if for each $(E,d) \in B^N$, each bijection f from N into itself, and each $i \in N$, we have $\mathcal{R}_i(E,d) = \mathcal{R}_{f(i)}\big(E, (d_{f(i)})_{i\in N}\big)$.
- *order preservation*, if for each $(E,d) \in B^N$ and each $\{i,j\} \subset N$, if $d_i \leq d_j$ we have $\mathcal{R}_i(E,d) \leq \mathcal{R}_j(E,d)$ and $d_i - \mathcal{R}_i(E,d) \leq d_j - \mathcal{R}_j(E,d)$.
- *claim monotonicity*, if for each $(E,d) \in B^N$, each $i \in N$, and each $d'_i \geq d_i$, we have $\mathcal{R}_i(E,(d_{-i},d'_i)) \geq \mathcal{R}_i(E,d)$.
- *linked claim-endowment monotonicity*, if for each $(E,d) \in B^N$, each $i \in N$ and each $\delta > 0$, we have $\mathcal{R}_i\big(E+\delta, (d_{-i}, d_i+\delta)\big) - \mathcal{R}_i(E,d) \leq \delta$.
- *other-regarding claim monotonicity*, if for each $(E,d) \in B^N$, each $i \in N$, and each $d'_i \geq d_i$, we have $\mathcal{R}_j(E,(d_{-i},d'_i)) \leq \mathcal{R}_j(E,d)$ for all $j \in N$, $j \neq i$.
- *endowment monotonicity*, if for each $(E,d) \in B^N$ and each $E' \geq 0$, if $d(N) \geq E' \geq E$ we have $\mathcal{R}(E',d) \geq \mathcal{R}(E,d)$.
- *homogeneity*, if for each $(E,d) \in B^N$ and each $\lambda > 0$, we have $\mathcal{R}(\lambda E, \lambda d) = \lambda \mathcal{R}(E,d)$.
- *composition down*, if for each $(E,d) \in B^N$ and each $E' < E$, we have $\mathcal{R}(E',d) = \mathcal{R}(E', \mathcal{R}(E,d))$.
- *composition up*, if for each $(E,d) \in B^N$ and each $E' \geq 0$, if $d(N) \geq E' > E$ we have $\mathcal{R}(E',d) = \mathcal{R}(E,d) + \mathcal{R}(E'-E, d - \mathcal{R}(E,d))$.
- *continuity*, if for each sequence $\{(E^\nu, d^\nu)\}$ of elements of B^N and each $(E,d) \in B^N$, if $\{(E^\nu, d^\nu)\}$ converges to (E,d) we have $\{\mathcal{R}(E^\nu, d^\nu)\}$ converges to $\mathcal{R}(E,d)$.
- *self-duality*, if for each $(E,d) \in B^N$ we have $\mathcal{R}(E,d) = d - \mathcal{R}\big(d(N) - E, d\big)$.

Some of these properties are stronger requirements than others. For instance, min-of-claim-and-equal-division lower bounds on awards implies $\frac{1}{|N|}$-truncated-claims lower bounds on awards, and other-regarding claim monotonicity implies claim monotonicity. Also, anonymity and order preservation are both stronger than equal treatment of equals. Moreover, composition down and composition up are both stronger than endowment monotonicity.

With each rule $\mathcal{R}$ we can associate a unique dual rule $\mathcal{R}^*$, the one defined by the right-hand side of the expression in the statement of the self-duality property: $\mathcal{R}^*(E,d) = d - \mathcal{R}\big(d(N) - E, d\big)$. A rule $\mathcal{R}$ is self-dual if $\mathcal{R} = \mathcal{R}^*$. Two properties are dual if whenever a rule satisfies one of them then its dual satisfies the other.

The following are pairs of dual properties: claims truncation invariance and minimal rights first; composition down and composition up; and claim monotonicity and linked claim-endowment monotonicity.

We also consider situations in which the population of claimants involved may vary. In this case, a bankruptcy problem is defined by first specifying $N \in \mathcal{N}$, then a pair $(E,d) \in B^N$. We still denote the class of all problems with claimant set N by B^N. So, a rule is a function defined on $\bigcup_{N\in\mathcal{N}} B^N$ that associates with each $N \in \mathcal{N}$ and each $(E,d) \in B^N$ an awards vector for (E,d). We say that a rule $\mathcal{R}$ satisfies:

- *population monotonicity*, if for each pair $\{N,N'\} \subset \mathcal{N}$ such that $N' \subset N$, and each $(E,d) \in B^N$ we have $\mathcal{R}_{N'}(E,d) \leq \mathcal{R}(E,d_{N'})$.
- *consistency*, if for each pair $\{N,N'\} \subset \mathcal{N}$ such that $N' \subset N$, and each $(E,d) \in B^N$ if $x = \mathcal{R}(E,d)$ we have $x_{N'} = \mathcal{R}\big(x(N'), d_{N'}\big)$. *Bilateral consistency* is the weaker property obtained by considering only subgroups of two remaining agents, that is, when $|N'| = 2$.
- *average consistency*, if for each $N \subset \mathcal{N}$, each $(E,d) \in B^N$, and each $i \in N$ if $x = \mathcal{R}(E,d)$ we have $x_i = \frac{1}{|N|-1} \sum_{j\in N\setminus\{i\}} \mathcal{R}_i(x_i + x_j, (d_i, d_j))$.

A coalitional game is an ordered pair (N, v) where $N \in \mathcal{N}$ is a finite set of players and $v : 2^N \to \mathbb{R}$, the characteristic function, satisfies $v(\emptyset) = 0$. A subset $S \in 2^N$ of N is referred to as a coalition. In general, $v(S)$ represents the joint payoff that can be obtained by the members of coalition S if they cooperate. For simplicity, we will write $v(i)$ instead of $v(\{i\})$ for $i \in N$. Let G^N be the set of all coalitional games with player set N. Given $S \in 2^N$, let $|S|$ be the number of players in S. The main focus within a cooperative setting is on how to share among the players the amount $v(N)$, the total joint payoff. Given a game $v \in G^N$, an allocation $x \in \mathbb{R}^N$ is said to be efficient if $x(N) = v(N)$. The set of all efficient allocations for a game $v \in G^N$ is the hyperplane $H(v) = \{x \in \mathbb{R}^N : x(N) = v(N)\}$. A solution is a mapping that associates with each game v in some admissible class an efficient payoff vector.

The set of imputations of a game $v \in G^N$ is defined as $I(v) = \big\{x \in H(v) : x_i \geq v(i) \text{ for all } i \in N\big\}$. The core is the set $C(v) = \big\{x \in I(v) : x(S) \geq v(S) \text{ for all } S \in 2^N\big\}$. The allocations that belong to the core are called stable allocations. A game $v \in G^N$ is called balanced if its core is non-empty. A game $v \in G^N$ is additive if $v(S) = \sum_{i\in S} v(i)$ for all $S \in 2^N$, in which case $C(v) = I(v) = \{\big(v(i)\big)_{i\in N}\}$. Thus, an additive game $v \in G^N$ is characterized by the vector $a = \big(v(i)\big)_{i\in N} \in \mathbb{R}^N$. To simplify the notation, we will usually identify by the same letter both the vector and the additive game. A game $v \in G^N$ is zero-normalized if $v(i) = 0$ for each player $i \in N$. Given a game $v \in G^N$ the zero-normalization of v is the game $v_0 \in G^N$ defined by $v_0(S) = v(S) - \sum_{i\in S} v(i)$, $S \in 2^N$. Clearly, $v = a + v_0$ where $a = \big(v(i)\big)_{i\in N}$. In particular, if $v \in G^N$ is an arbitrary balanced game then v_0 is also balanced and $C(v) = a + C(v_0)$.

When the core of a game is non-empty, if one considers that all of the core alternatives are equally preferable then selecting the average stable payoff seems to be an intuitive and natural choice. Given a balanced game $v \in G^N$, González-Díaz and Sánchez-Rodríguez (2007) define the core-center $\mu(v)$ as the mathematical expectation of the uniform distribution over the core of the game, i.e., the center of gravity (centroid) of $C(v)$. In general, given a convex polytope $K \subset H(v)$ denote by $\text{Vol}_{n-1}(K)$, or simply $\text{Vol}(K)$ if no confusion is possible, its $(n-1)$-dimensional Lebesgue measure and by $\mu(K)$ its center of gravity. In particular, $\mu(v) = \mu(C(v))$. Convexity of K ensures that $\mu(K) \in K$. Also $\mu(a + K) = a + \mu(K)$ for all $a \in \mathbb{R}^N$. Therefore, if $v \in G^N$ is a balanced game, $v_0 \in G^N$ is its zero-normalization, and $a = \big(v(\{i\})\big)_{i\in N}$ we have that $\mu(v) = a + \mu(v_0)$. We will make extensive use of the following property: if $K = K_1 \cup K_2$, $\text{Vol}\big(K_1 \cap K_2\big) = 0$, and $\rho = \frac{\text{Vol}(K_1)}{\text{Vol}(K)}$, then $\mu(K) = \rho\mu(K_1) + (1-\rho)\mu(K_2)$.

The bankruptcy game $v \in G^N$ associated with the bankruptcy problem $(E,d) \in B^N$ is defined by $v(S) = \max\big\{0,\ E - d(N\setminus S)\big\}$ for all $S \in 2^N$. In general, given a bankruptcy problem $(E,d) \in B^N$ and its associated bankruptcy game $v \in G^N$ we will use the notations $H(E,d) = H(v)$, $I(E,d) = I(v)$, and $C(E,d) = C(v)$. Bankruptcy games are balanced so $C(E,d) \neq \emptyset$ for all $(E,d) \in B^N$. Let $(E,d) \in B^N$ be a bankruptcy problem, $v \in G^N$ the associated bankruptcy game, and $a = \big(v(i)\big)_{i\in N}$, then the bankruptcy game $v_0 \in G^N$ associated with the bankruptcy problem $\big(E - a(N), d - a\big) \in B^N$ is the zero-normalization of v.

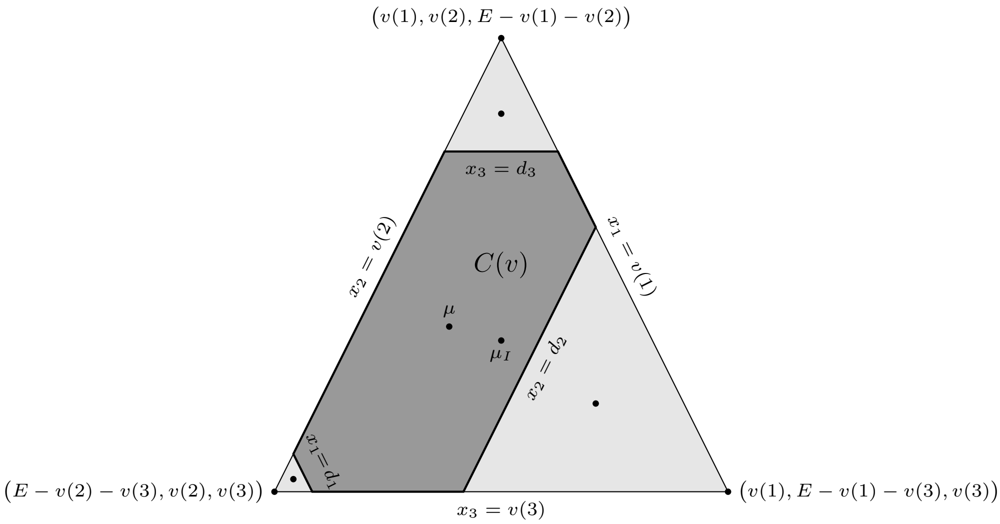

Figure 1: The core of a 3-player bankruptcy game.

3 The core of a bankruptcy game

We devote this section to analyze in detail the core of the bankruptcy game associated with a bankruptcy problem. Throughout the paper, we will assume, without loss of generality, that $N = \{1, \ldots, n\}$.

Let $(E, d) \in B^N$ be a bankruptcy problem and $v \in G^N$ the associated bankruptcy game. First, note that the minimal right of claimant $i \in N$ is $m_i(E, d) = v(i)$ and the truncated claim of claimant $i \in N$ is $t_i(E, d) = v(N) - v(N \backslash \{i\})$. Bankruptcy games ara balanced, so $C(E, d)$ is a non-empty convex polytope contained in the efficiency hyperplane and therefore it has, at most, dimension $n - 1$. Moreover, the core of a bankrutpcy game is heavily structured. Curiel et al. (1987) showed that it coincides with the Weber set (Weber, 1988) and with the core-cover (Tijs and Lipperts, 1982). Dietzenbacher (2018) showed that it coincides with the reasonable set (Gerard-Varet and Zamir, 1987). In fact, the core of a bankruptcy game consists of all efficient allocations which are bounded from below by the minimal rights and bounded from above by the truncated claims.

$$C(E, d) = \big\{x \in H(E, d) : m_i(E, d) \leq x_i \leq t_i(E, d) \text{ for all } i \in N\big\}.$$

With the help of a simple diagram, we will try to explain the intuition behind our analysis. Figure 1 shows a sketch of the imputation set of a generic 3-claimant bankruptcy game and a core with the maximum number of extreme points. We explore in the Appendix the structure of the imputation set of a bankruptcy game and we show, see Lemma A.1, that if all the claims are bigger than the endowment then the core coincides with the imputation set. If that is the case, the most notable game theoretical solutions agree to recommend the egalitarian division among the claimants: the barycenter of the imputation triangle, $\mu_I = \mu(I(E, d))$. But, if at least one individual claim is less than the endowment then we have imputations that are not stable. Figure 1 depicts the imputation set and the core of a bankruptcy game when $d_i < E$ for all $i \in N$ and $d_j + d_k < E$ for all $j \neq k$. Then, the imputation set is the union of the core and three equilateral triangles. Each imputation in the triangle at bottom left, U_1, awards claimant 1 with at least d_1. Let us assume that 2% of the imputations generously reward the first claimant with at least her claim. Analogously, we can identify the sets of imputations, U_2 (bottom right triangle) and U_3 (upper triangle), that reward the second and third claimant with at least d_2 and d_3 respectively. Say that they represent 30% and 10% of the imputations, so that the remaining 58% of the imputations are stable. The core-center rule recommends the unique core allocation for which the initial egalitarian division is the weighted average of the egalitarian selections inside the regions that benefit each claimant and the core-center selection. As this example illustrates, ratios of volumes like $\frac{\text{Vol}(C(E,d))}{\text{Vol}(I(E,d))}$, the percentage of imputations that are stable, play an special role when dealing with the core-center rule, so we must put special emphasis on studying the volume of the core of a bankruptcy game. Let $d = (d_1, \ldots, d_n) \in \mathbb{R}^N$ be a sorted vector of claims in ascending order, i.e., $0 < d_1 \leq \cdots \leq d_n$. We define the volume function $V(\cdot, d) \colon [0, d(N)] \to \mathbb{R}$ that assigns to each $E \in [0, d(N)]$ the volume of the core of the bankruptcy game associated with the bankruptcy problem $(E, d) \in B^N$, that is, $V(E, d) = \text{Vol}_{n-1}(C(E, d))$.

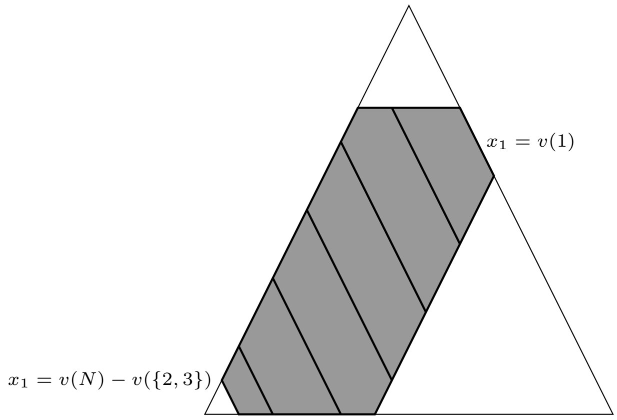

Figure 2: The core of a bankruptcy game and the cores of some reduced problems v^1_x.

In the Appendix, we carry out an extensive analysis of the function $V(\cdot, d)$. If $|N| > 2$, we show that $V(\cdot, d)$ is differentiable and, by computing its derivative, we examine how the core of the bankruptcy game changes with respect to variations on the endowment.

On the other hand, the region U_1, for instance, is related to the situation that arises when the first claimant leaves with her best award d_1, which leads us to study the reduced games introduced by Davis and Maschler (1965). Let $v \in G^N$, $i \in N$, and $x \in \mathbb{R}^N$, the max reduced game $v^i_x \in G^{N\setminus\{i\}}$ is define as:

$$v^i_x(S) = \begin{cases} 0 & \text{if } S = \emptyset \\ v(N) - x_i & \text{if } S = N\setminus\{i\} \\ \max\{v(S \cup \{i\}) - x_i, v(S)\} & \text{otherwise} \end{cases}$$

The player set of v^i_x is obtained by removing player i from the original player set N. In the reduced game, the players of $N\setminus\{i\}$ consider how to divide the total amount assigned to them by x assuming that player i gets exactly x_i. Then, the worth of $N\setminus\{i\}$ is $x(N\setminus\{i\}) = v(N) - x_i$. Funaki and Yamato (2001) showed that the core satisfies a consistency property: if $v \in G^N$, $i \in N$, and $x \in C(v)$ then $v^i_x \in G^{N\setminus\{i\}}$ is a balanced game and $x_{-i} \in C(v^i_x)$. As a consequence, if we denote $I_i = [v(i), v(N) - v(N\setminus\{i\})]$ then

$$C(v) = \bigcup_{x_i \in I_i} \{x_i\} \times C(v^i_x).$$

This decomposition is illustrated in Figure 2, where each line inside the core of the bankruptcy game is the core of a reduced problem v^1_x for some $x \in \mathbb{R}^3$ such that $v(1) \le x_1 \le v(N) - v(\{2,3\})$. Aumann and Maschler (1985) showed that the max reduced bankruptcy game is the game corresponding to the “reduced bankruptcy problem”. Let $(E, d) \in B^N$ be a bankruptcy problem, $v \in G^N$ the associated bankruptcy game, $i \in N$, and $x \in C(E, d)$. Then $v^i_x \in G^{N\setminus\{i\}}$ is the bankruptcy game associated to the bankruptcy problem $(E - x_i, d_{-i}) \in B^{N\setminus\{i\}}$. The following result is then straightforward.

Proposition 3.1. *Let $(E, d) \in B^N$ be a bankruptcy problem, $i \in N$, and $I_i = [m_i(E, d), t_i(E, d)]$. Then*

$$C(E, d) = \bigcup_{x_i \in I_i} \{x_i\} \times C\big(E - x_i, d_{-i}\big).$$

Given a bankruptcy problem $(E, d) \in B^N$ consider the, so called, dual problem $(d(N) - E, d) \in B^N$ and let $v^* \in G^N$ be the associated bankruptcy game. It turns out that the core of a bankruptcy game is self-dual: the core of v is the translate of the minus core of v^* by the vector of claims d.

Proposition 3.2. *Let $(E, d) \in B^N$ be a bankruptcy problem. Then $C(E, d) = d - C\big(d(N) - E, d\big)$.*

Proof. Let $v \in G^N$ and $v^* \in G^N$ be the games associated with the problems $(E, d) \in B^N$ and $(d(N) - E, d) \in B^N$ respectively. Then,

$$C\big(d(N) - E, d\big) = \big\{y \in \mathbb{R}^N : y(N) = d(N) - E,\ v^*(i) \le y_i \le \min\{d(N) - E, d_i\} \text{ for all } i \in N\big\}.$$

Observe that $v^*(i) = \max\{0, d_i - E\}$, $d_i - \min\{d(N) - E, d_i\} = v(i)$ and $d_i - v^*(i) = \min\{E, d_i\}$ for all $i \in N$. Therefore, for each $i \in N$, we have that $v(i) \leq x_i \leq \min\{E, d_i\}$ if and only if $v^*(i) \leq d_i - x_i \leq \min\{d(N) - E, d_i\}$. Now, the result is straightforward. □

Since the core is self-dual the graph of the volume function $V(\cdot, d)$ is symmetric with respect to the axis $E = \frac{1}{2}d(N)$ determined by the half sum of the claims (see Figure 9).

4 Basic properties of the core-center rule

One way of defining meaningful rules is by applying game theoretical solutions to the bankruptcy game. Consider the function $\mu : B^N \to \mathbb{R}^N$ that assigns to each bankruptcy problem $(E, d) \in B^N$ the core-center of its associated bankruptcy game $v \in G^N$, that is, $\mu(E, d) = \mu(v) = \mu(C(v)) \in C(v)$. So, following the game theoretical approach, we provide a new and intuitive rule for bankruptcy problems: the core-center rule. We aimed to show that the core-center rule is a well behaved rule, that is, it satisfies a good number of properties. In this section, we study a set of the basic properties that most of the standard bankruptcy rules satisfy.

Proposition 4.1. *The core-center $\mu : B^N \to \mathbb{R}^N$ is a rule that satisfies minimal rights first, equal treatment of equals, anonymity, homogeneity, continuity, self-duality, and claims truncation invariance.*

Proof. Let $(E, d) \in B^N$ be a bankruptcy problem and $v \in G^N$ the associated bankruptcy game. We know that if $v_0 \in G^N$ is the zero-normalization of v then $\mu(v) = a + \mu(v_0)$, where $a = \big(v(\{i\})\big)_{i \in N}$. Therefore, μ satisfies minimal rights first. González-Díaz and Sánchez-Rodríguez (2007) showed that the core-center, as a solution defined in the class of balanced games, treats symmetric players equally and that it is a homogeneous and continuous function of the values of the characteristic function. In the context of bankruptcy problems, equal treatment of equals and anonymity hold because agents with the same claims are symmetric players in the associated bankruptcy game. Since, given $\lambda > 0$ and $S \in 2^N$, $v_\lambda(S) = \lambda v(S)$, where v_λ is the bankruptcy game associated with $(\lambda E, \lambda d) \in B^N$, the core-center rule satisfies homogeneity. The values, $v(S)$, $S \in 2^N$, of the characteristic function are continuous with respect to the endowment E and the claims d. Then, the core-center rule satisfies continuity because it is a composition of continuous functions. Self-duality is a consequence of Proposition 3.2. Claims truncation invariance and minimal rights first are dual properties. Since the core-center rule is self-dual and satisfies minimal rights first then it satisfies claims truncation invariance. □

Suppose that there are only two claimants, so let $N = \{1, 2\}$ and $d = (d_1, d_2) \in \mathbb{R}^N$, with $0 \leq d_1 \leq d_2$. Then, $C(E, d) = I(E, d)$ is the line segment with endpoints $(v(1), E - v(1))$ and $(E - v(2), v(2))$, where $v(1) = \max\{0, E - d_2\}$ and $v(2) = \max\{0, E - d_1\}$ (see Figure 3). The core-center rule is the middle point of this segment,

$$\mu(E, d) = \begin{cases} \left(\frac{E}{2}, \frac{E}{2}\right) & \text{if } 0 \leq E \leq d_1 \\ \left(\frac{d_1}{2}, E - \frac{d_1}{2}\right) & \text{if } d_1 \leq E \leq d_2 \\ \left(\frac{E + d_1 - d_2}{2}, \frac{E - d_1 + d_2}{2}\right) & \text{if } d_2 \leq E \leq d_1 + d_2 \end{cases}. \tag{1}$$

Therefore, the core-center rule satisfies the contested garment principle (Aumann and Maschler, 1985) or concede-and-divide principle. That means that the core-center rule coincides with the Talmud rule (Aumann and Maschler, 1985) and the random arrival rule (O'Neill, 1982) for bankruptcy problems with two claimants. Since concede-and-divide violates composition down so does the core-center rule.

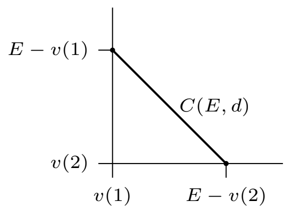

Figure 3: The core of a bankruptcy game with two claimants.

Let us illustrate with a simple example that the core-center rule differs from the standard bankruptcy rules. It also manifests that the core-center rule fails min-of-claim-and-equal-division lower bounds on awards.

Example 4.2. *Let $N = \{1,2,3\}$ and consider the bankruptcy problem $(E,d) \in B^N$ with $E = 3$ and $d = (1,2,2)$. The next table displays the awards assigned to this particular problem by the core-center rule (μ), the proporcional rule (PRO), the constrained equal awards rule (CEA), the constrained equal losses rule (CEL), the Talmud rule (T), and the random arrival rule (RA). The computation of the core-center rule is detailed in Example B.3.*

CEA	*CEL*	*PRO*	*T*	*RA*	μ
$(1,1,1)$	$(\frac{1}{3},\frac{4}{3},\frac{4}{3})$	$(\frac{3}{5},\frac{6}{5},\frac{6}{5})$	$(\frac{1}{2},\frac{5}{4},\frac{5}{4})$	$(\frac{2}{3},\frac{7}{6},\frac{7}{6})$	$(\frac{5}{9},\frac{11}{9},\frac{11}{9})$

Observe that $t\big(\frac{E}{3},d\big) = (1,1,1)$ and $\mu_1(E,d) < 1$, so the core-center rule violates min-of-claim-and-equal-division lower bounds on awards.

Dagan (1996) shows that the constrained equal awards rule is the only rule satisfying equal treatment of equals, claims truncation invariance and composition up. Then, the core-center rule does not satisfy composition up. Composition down and composition up are dual properties. Then the core-center rule violates composition down. Moreno Ternero and Villar (2004) prove that the Talmud rule is the only rule satisfying claims truncation invariance, self-duality an bilateral consistency. Then the core-center rule fails consistency.

Since the core-center rule satisfies anonymity, in what follows we will assume that given a bankruptcy problem $(E,d) \in B^N$, the vector of claims $d = (d_1,\ldots,d_n) \in \mathbb{R}^N$ is sorted in ascending order, i.e., $d_1 \leq \cdots \leq d_n$.

Suppose that, given a bankruptcy problem $(E,d) \in B^N$, the core of the bankruptcy game $C(E,d)$ can be decomposed as the union of two pieces, C_1 and C_2, with negligible intersection, i.e., $C(E,d) = C_1 \cup C_2$ and $\mathrm{Vol}(C_1 \cap C_2) = 0$. In some instances, C_1 and C_2 are the cores, possibly translated by a specific vector, of some particular bankruptcy games obtained from (E,d). Whenever this is the case, if $p = \frac{\mathrm{Vol}(C_1)}{V(E,d)}$ is the percentage of stable allocations that belong to C_1, then $1-p = \frac{\mathrm{Vol}(C_2)}{V(E,d)}$ is the percentage of stable allocations that belong to C_2, and the core-center rule $\mu(E,d)$ is the weighted average of the core-centers of the pieces, $\mu(E,d) = p\mu(C_1) + (1-p)\mu(C_2)$. In the Appendix we present two such decompositions that provide a geometrical interpretation in terms of the core of the bankruptcy game of order preservation and claim monotonicity respectively.

Order preservation is, in fact, the conjunction of two dual properties: order preservation in awards and order preservation in losses. First we show that the awards recommended by the core-center rule are ordered as claims are.

Proposition 4.3. *Let $(E,d) \in B^N$ and $i \in N\backslash\{n\}$. If $d_i \leq d_{i+1}$ then $\mu_i(E,d) \leq \mu_{i+1}(E,d)$.*

Proof. Since the core-center rule satisfies equal treatment of equals we assume that $d_i < d_{i+1}$. If $d_i \geq E$ then, by Proposition B.1, $\mu_i(E,d) = \mu_{i+1}(E,d)$. Let $d_i < E$. If $m(E,d) = 0$, we can apply Proposition C.1. Denote $e^i \in \mathbb{R}^N$ the vector with 1 in the ith-coordinate and 0's elsewhere. Let $a = (d_{-(i+1)}, d_i)$, $b = d_i e^{i+1}$, $c = d - b$, and $\rho = \frac{V(E,a)}{V(E,d)}$. Since μ satisfies equal treatment of equals, $\mu_{i+1}(E,a) = \mu_i(E,a)$. Moreover, $\mu_{i+1}\big(b + C(E-d_i,c)\big) = d_i + \mu_{i+1}(C(E-d_i,c)) \geq d_i \geq \mu_i(C(E-d_i,c)) = \mu_i\big(b + C(E-d_i,c)\big)$. Then,

$$\mu_{i+1}(E,d) = \rho\mu_{i+1}(C(E,a)) + (1-\rho)\mu_{i+1}\big(b + C(E-d_i,c)\big) \geq \rho\mu_i(E,a) + (1-\rho)\mu_i(E-d_i,c) = \mu_i(E,d).$$

In general, if $v \in G^N$ is the bankruptcy game associated with $(E,d) \in B^N$, we know that $v = m(E,d) + v_0 = \big(v(i)\big)_{i\in N} + v_0$ where $v_0 \in G^N$ is the zero-normalization of v. Then, since $v(i) \leq v(i+1)$, we have that $\mu_{i+1}(E,d) = v(i+1) + \mu_{i+1}(v_0) \geq v(i) + \mu_i(v_0) = \mu_i(E,d)$. □

Order preservation in awards and order preservation in losses are dual properties. So, the losses implied by the core-center rule are also ordered as claims are.

Proposition 4.4. *The core-center rule satisfies order preservation.*

Next, we show that the core-center rule satisfies claim monotonicity, that is, if some agent's claim increases then her award should not decrease. The dual property of claim monotonicity is linked claim-endowment monotonicity that states that if an agent's claim and the endowment increase by equal amounts (their changes are linked) then this claimant's award should increase by at most this amount.

Proposition 4.5. *The core-center rule satisfies claim monotonicity and linked claim-endowment monotonicity.*

Proof. Let $(E,d) \in B^N$ be a bankruptcy problem, $v \in G^N$ the associated bankruptcy game and $i \in N\backslash\{n\}$. Let $d_i \leq d'_i \leq d_{i+1}$. If $d_i \geq E$ then $C(E,d') = C(E,d)$ and $\mu_i(E,d') = \mu_i(E,d)$. Assume that $d_i < E$ so that Proposition C.2 holds. Let $e^i \in \mathbb{R}^N$ be the vector with 1 in the ith-coordinate and 0's elsewhere, $d' = (d_{-i}, d'_i)$, $b = d_i e^i$, $c = d' - b$, and $\rho = \frac{V(E,d)}{V(E,d')}$. Clearly, $\mu_i(E,d) \leq d_i + \mu_i(E - d_i, c)$. Then, $\mu_i(E,d') = \rho\mu_i(E,d) + (1-\rho)\big(d_i + \mu_i(E - d_i, c)\big) \geq \mu_i(E,d)$. The core-center rule also satisfies linked claim-endowment monotonicity, since this property and claim monotonicity are dual properties. □

5 Consistency and endowment differentiability

Recall that a rule $\mathcal{R}$ satisfies consistency if for each pair $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$, and each $(E,d) \in B^N$ if $x = \mathcal{R}(E,d)$ then $x_{N'} = \mathcal{R}\big(x(N'), d_{N'}\big)$. Consider a bankruptcy problem and apply a rule to it. Imagine that some claimants leave with their awards. The remaining claimants, the members of N', faced the reduced problem relative to the subgroup and the initial recommendation: their claims are unchanged and the endowment is the difference between the amount initially available and the sum of the awards to the agents who left, $x(N') = E - x(N\backslash N')$. The rule is consistent if it awards to each of the claimants in N' the same amount as it did initially. Aumann and Maschler (1985) showed that the Talmud rule is the only rule to agree with concede-and-divide for two claimants and to be bilaterally consistent. Then, as we have already seen, the core-center rule is not consistent. Naturally, weaker consistency requirements have been proposed in the literature that allow us to extend a two-claimant rule to arbitrary populations, for instance, average consistency. Dagan and Volij (1997) show that if the initial two-claimant rule happens to have a consistent extension, then the consistent-on-average rule is precisely this consistent extension. Therefore, the core-center rule does not satisfy average consistency. Here, we prove that the core-center rule satisfies a consistency-type property: the awards chosen by the core-center rule for a problem are the weighted average of the choices made by the rule in each of the reduced problems that result when a claimant receives any award in the core of the bankruptcy game and leaves.

If claimant $i \in N$ gets the minimum right $m_i(E,d)$ then the remainder $R_i(E,d) = E - m_i(E,d) = \min\{E, d(N\backslash\{i\})\} = v(N) - v(i)$ is the maximum award that the other claimants, the members of $N\backslash\{i\}$, together can get. Analogously, if claimant $i \in N$ gets the truncated claim $t_i(E,d)$ then the remainder $r_i(E,d) = E - t_i(E,d) = \max\{0, E - d_i\} = v(N\backslash\{i\})$ is the minimum award that the other claimants can get. Now, assume that agent i receives an award x_i such that $m_i(E,d) \leq x_i \leq t_i(E,d)$ and leaves. The remaining agents face the reduced problem $(u, d_{-i}) \in B^{N\backslash\{i\}}$ where $u = E - x_i$ and $r_i(E,d) \leq u \leq R_i(E,d)$. But, see Proposition 3.1, the core $C(u, d_{-i})$ is a cross section of the core of the initial problem $C(E,d)$. Naturally, this cross section might be "bigger" or "smaller" depending on the value of u. Therefore, the relative measure of stable allocations available to the remaining agents, $\frac{V(u,d_{-i})}{V(E,d)}$, varies with u. For each $i \in N$ consider the function $g_i\colon (0, d(N)) \times [0, d(N\backslash\{i\})] \longrightarrow \mathbb{R}$ defined as:

$$g_i(E,u) = \frac{\sqrt{n}}{\sqrt{n-1}} \frac{V(u,d_{-i})}{V(E,d)}, \text{ for all } (E,u) \in (0,d(N)) \times [0, d(N\backslash\{i\})].$$

Clearly, $g_i(E,u) \geq 0$ for all $(E,u) \in (0,d(N)) \times [0, d(N\backslash\{i\})]$. Therefore, $g(E,\cdot)$ is a weight function for all $E \in (0, d(N))$ that, as we prove next, is a probability density function on the interval $[r_i(E,d), R_i(E,d)]$. The core-center rule selects for each claimant in $N\backslash\{i\}$ the weighted average of the choices made by the rule over all of the reduced problems where agent i receives and award between her minimal right and her truncated claim and leaves.

Theorem 5.1. *Assume that* $|N| \geq 3$. *If* $(E,d) \in B^N$ *and* $i \in N$ *then* $\int_{r_i(E,d)}^{R_i(E,d)} g_i(E,u)du = 1$. *Moreover,*

$$\mu_j(E,d) = \int_{r_i(E,d)}^{R_i(E,d)} \mu_j(u, d_{-i}) g_i(E,u)du \textit{ for all } j \in N\backslash\{i\}.$$

Proof. The result follows from Theorem D.1 in the Appendix by applying the change of variable $u = E - s$. □

The property stated in Theorem 5.1 implicitly defines a mechanism to extend the concede-and-divide rule to an arbitrary population of claimants: take the weighted average of the amounts awarded by the rule in the reduced problems obtained when a single-claimant leaves. We say that a rule satisfies single-agent weighted consistency if for each pair of claimants $\{i,j\} \subset N$, claimant j's award is a weighted average of the recommendations made by the rule in all the reduced problems with respect to $\{i\}$ and any vector x in the core of the bankruptcy game.

- A rule $\mathcal{R}$ satisfies *single-agent weighted consistency* if, for each $(E,d) \in B^N$ and each $i \in N$, there exists a weight function $w_i(E,\cdot)$, for all $E \in (0, d(N))$, such that, for each $j \in N \backslash \{i\}$ we have

$$\mathcal{R}_j(E,d) = \int_{r_i(E,d)}^{R_i(E,d)} \mathcal{R}_j(u, d_{-i}) w_i(E,u) du.$$

where $r_i(E,d) = E - t_i(E,d)$ and $R_i(E,d) = E - m_i(E,d)$.

As a corollary of Theorem 5.1 we have that the core-center rule satisfies single-agent weighted consistency when $|N| \geq 3$. In fact, the weight functions are given by the relative measure of stable allocations that belong to the core of the reduced games. These weight functions are the marginal density functions of the uniform distribution over the core. Therefore, the following characterization holds.

Corollary 5.2. *The core-center rule is the only rule to agree with concede-and-divide for two claimants and to satisfy single-agent weighted consistency.*

Naturally, a way to define meaningful weight functions is to consider different probability distributions over the core of the game that could lead to extensions of two claimant rules. In the present formulation, the single-agent weighted consistency property is a strong requirement, but it opens the door to explore weaker forms. A discrete version, when considering finite probability distribution over the core, could be related with the average consistency property.

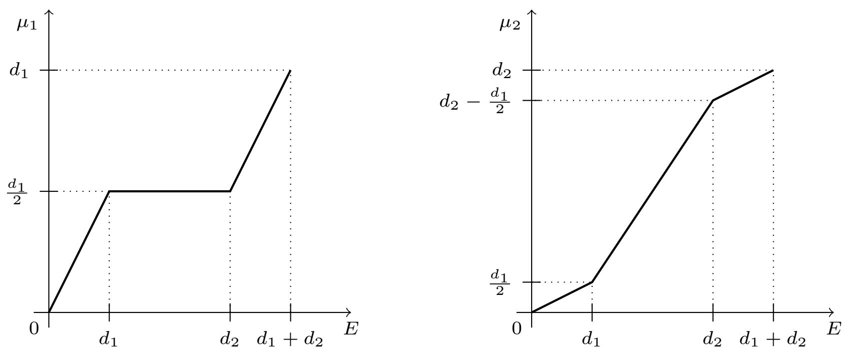

Figure 4: Awards chosen by the concede-and-divide rule as a function of E.

Given a rule $\mathcal{R}$ and a vector of claims $d \in \mathbb{R}^N$, the path followed by the awards vector chosen by $\mathcal{R}$ as the endowment increases from 0 to $d(N)$, that is, the function $\mathcal{R}(\cdot, d)\colon [0, d(N)] \to \mathbb{R}^N$, is called the path of awards of the rule for the claims vector. A rule $\mathcal{R}$ satisfies endowment continuity if the path of awards of the rule is continuous for all claims vector. Endowment continuity is a weaker property than continuity. Certainly, the core-center rule, being continuous, satisfies endowment continuity. A stronger property, endowment differentiability, is to require the paths of awards to be differentiable. As Figure 4 illustrates, for bankruptcy problems with two claimants, the concede-and-divide rule, and therefore the core-center rule, violates endowment differentiability. Nevertheless, as a first implication of Theorem 5.1, we prove in the Appendix, see Theorem D.2, that the core-center rule is endowment differentiable for problems with more than two claimants.

Proposition 5.3. *If $|N| \geq 3$ then the core-center rule satisfies endowment differentiability.*

The differentiability of the path of awards of the core-center rule is illustrated in Figure 5 and Figure 6. Among the standard bankruptcy rules, and for an arbitrary set of claimants N, the proportional rule satisfies endowment differentiability but the constraint equal awards rule, the constraint equal losses rule, the Talmud rule, and the random arrival rule violate it.

6 Endowment monotonicity and related properties

Endowment monotonicity states that when there is more to be divided then nobody should lose. An endowment continuous rule $\mathcal{R}$ satisfies endowment monotonicity if and only if $\mathcal{R}_i(E,d)$, the award chosen by the rule for claimant i as a function of the endowment, is monotonically increasing on $[0,d(N)]$ for all claims vector d. The concede-and-divide rule, whose awards for each claimant are the continuous piecewise linear functions depicted in Figure 4, satisfies endowment monotonicity.

Assume that $|N|\geq 3$ and that $d=(d_1,\dots,d_n)\in\mathbb{R}^N$ with $0\leq d_1\leq\cdots\leq d_n$. Since the core-center rule is endowment differentiable whenever $|N|\geq 3$, in order to prove that it satisfies endowment monotonicity we have to show that $\frac{\partial\mu_j}{\partial E}(E,d)\geq 0$ for all $E\in[0,d(N)]$ and for all $j\in N$. The derivatives $\frac{\partial\mu_j}{\partial E}$, $j\in N$, are computed in Theorem D.2 in the Appendix.

Theorem 6.1. *The core-center rule satisfies endowment monotonicity. Moreover, if $(E,d)\in B^N$ is a bankruptcy problem then $\mu(r_i(E,d),d_{-i})\leq\mu_{N\backslash\{i\}}(E,d)\leq\mu(R_i(E,d),d_{-i})$ for all $i\in N$.*

Proof. Let $d=(d_1,\dots,d_n)\in\mathbb{R}^N$ such that $0<d_1\leq\cdots\leq d_n$. We proceed by induction on the number of claimants. We already know, that if $|N|=2$ then $\mu_j(\cdot,d)$ is monotonically increasing for all $j\in N$. Now, let $|N|\geq 3$, $i\in N$, and assume that $\mu_j(\cdot,d_{-i})$ is monotonically increasing for all $j\in N\backslash\{i\}$. According to Theorem D.2 it suffices to prove that $\mu_j(\cdot,d)$ is monotonically increasing on $\big[0,\min\big\{\frac{1}{2}d(N),d(N\backslash\{n\})\big\}\big]$ for all $j\in N$ or, equivalently, that $\frac{\partial\mu_j}{\partial E}(E,d)\geq 0$ for all $E\in\big[0,\min\big\{\frac{1}{2}d(N),d(N\backslash\{n\})\big\}\big]$ and all $j\in N$. By the induction hypothesis, $\mu_j(r_i(E,d),d_{-i})\leq\mu_j(u,d_{-i})\leq\mu_j(R_i(E,d),d_{-i})$ for all $u\in[r_i(E,d),R_i(E,d)]$ and all $j\in N\backslash\{i\}$. Therefore,

$$\int_{r_i(E,d)}^{R_i(E,d)}\mu_j(r_i(E,d),d_{-i})g_i(E,u)du\leq\int_{r_i(E,d)}^{R_i(E,d)}\mu_j(u,d_{-i})g_i(E,u)du\leq\int_{r_i(E,d)}^{R_i(E,d)}\mu_j(R_i(E,d),d_{-i})g_i(E,u)du$$

or, equivalently, from Theorem D.1, $\mu_j(r_i(E,d),d_{-i})\leq\mu_j(E,d)\leq\mu_j(R_i(E,d),d_{-i})$. From these inequalities and the expressions obtained in Theorem D.2 it follows that if $E\in\big[d_1,\min\big\{\frac{1}{2}d(N),d(N\backslash\{n\})\big\}\big]$ then $\frac{\partial\mu_n}{\partial E}(E,d)=g_1(E,E)\big(\mu_n(E,d_{-1})-\mu_n(E,d)\big)+g_1(E,E-d_1)\big(\mu_n(E,d)-\mu_n(E-d_1,d_{-1})\big)\geq 0$ and $\frac{\partial\mu_j}{\partial E}(E,d)=g_n(E,E)\big(\mu_j(E,d_{-n})-\mu_j(E,d)\big)+\chi_n(E,d)g_n(E,E-d_n)\big(\mu_j(E,d)-\mu_j(E-d_n,d_{-n})\big)\geq 0$ for all $j\in N\backslash\{n\}$. □

Observe that Theorem 6.1 provides some important bounds that relate the core-center rule of a bankruptcy problem with the core-center rule of the reduced problems obtained when an agent leaves with her best and worst awards.

Corollary 6.2. *Let $(E,d)\in B^N$ be a bankruptcy problem and $i\in N$. If $d_i<E\leq d(N\backslash\{i\})$ then $\mu(E-d_i,d_{-i})\leq\mu_{N\backslash\{i\}}(E,d)\leq\mu(E,d_{-i})$.*

Example 6.3. *Let $N=\{1,2,3\}$ and $d=(2,4,5)\in\mathbb{R}^N$ so that $d(N)=11$ and $d_3=5\leq\frac{1}{2}d(N)=5.5\leq d(N\backslash\{3\})=6$. Clearly $r_3(E,d)=\begin{cases}0 & \text{if } E\leq 5\\ E-5 & \text{if } E>5\end{cases}$ and $R_3(E,d)=\begin{cases}E & \text{if } E\leq 6\\ 6 & \text{if } E>6\end{cases}$. Therefore,*

$$\mu_1(r_3(E,d),d_{-3})=\begin{cases}0 & \text{if } E\leq 5\\ \frac{E-5}{2} & \text{if } 5<E\leq 7\\ 1 & \text{if } 7<E\leq 9\\ \frac{E-7}{2} & \text{if } 9<E\leq 11\end{cases}\qquad \mu_1(R_3(E,d),d_{-3})=\begin{cases}\frac{E}{2} & \text{if } E\leq 2\\ 1 & \text{if } 2<E\leq 4\\ \frac{E-2}{2} & \text{if } 4<E\leq 6\\ 2 & \text{if } 6<E\leq 11\end{cases}$$

The graphs of the coordinates $\mu_j(\cdot,d)$, $j=1,2,3$, are depicted in Figure 5. We can see that they are monotonically increasing and that $\mu_1(E,d)\leq\mu_2(E,d)\leq\mu_3(E,d)$ for all $E\in[0,11]$. Also, the self-duality property corresponds with the special symmetry of the graphs with respect to $E=5.5$. In Figure 5 right, we observe graphically that $\mu_1(r_3(E,d),d_{-3})\leq\mu_1(E,d)\leq\mu_1(R_3(E,d),d_{-3})$ for all $E\in[0,11]$.

Once endowment monotonicity of the core-center rule has been established we can address population monotonicity, other-regarding claim monotonicity and $\frac{1}{|N|}$-truncated-claims lower bounds on awards. Population monotonicity states that if the population of claimants enlarges but the amount to divide stays the same, each of the claimants initially present should receive at most as much as she did initially.

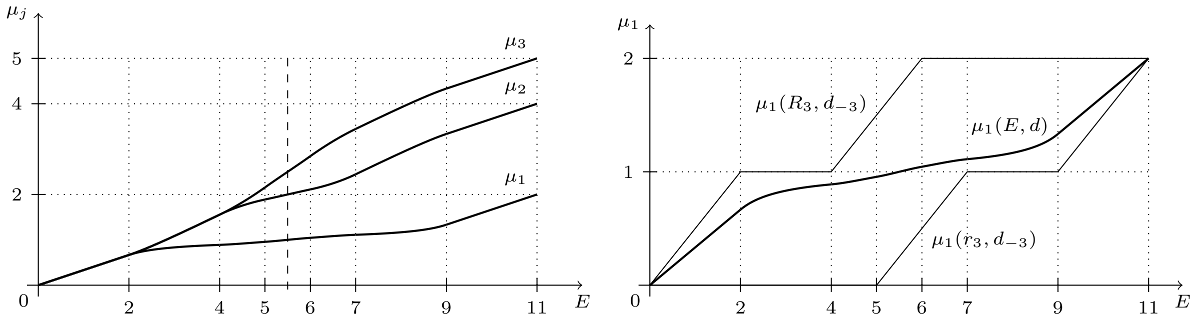

Figure 5: The core-center rule as a function of the endowment when $d = (2, 4, 5)$.

Proposition 6.4. *The core-center rule satisfies population monotonicity.*

Proof. Let $\{N, N'\} \subset \mathcal{N}$ such that $N' \subset N$. We have to prove that $\mu_{N'}(E, d) \leq \mu(E, d_{N'})$. First, let $N' = N \backslash \{i\}$ for some $i \in N$. Then, according to Corollary 6.2 and since μ satisfies endowment monotonicity, $\mu_{N'}(E, d) \leq \mu(R_i(E, d), d_{-i}) \leq \mu(E, d_{-i})$. The general case follows applying repeatedly this result. □

We know, see Proposition 4.5, that the core-center rule satisfies claim monotonicity, The property of other-regarding claim monotonicity considers the impact that an increase in some agent's claim has on the others. It requires that none of these claimants' awards should increase.

Proposition 6.5. *The core-center rule satisfies other-regarding claim monotonicity.*

Proof. Let $(E, d) \in B^N$ be a bankruptcy problem such that $0 \leq d_1 \leq \cdots \leq d_n$, $i \in N$, and $d_{i+1} \geq d'_i \geq d_i$. Denote $d' = (d_{-i}, d'_i) \in \mathbb{R}^N$. If $E \leq d_i$ then $C(E, d) = C(E, d')$ and $\mu(E, d) = \mu(E, d')$. Assume that $d_i < E$. Let $b = d_i e^i$, $c = d' - b$, $p = \frac{\text{Vol}(E,d)}{\text{Vol}(E,d')}$, and $j \in N \backslash \{i\}$. Then, from Proposition C.2, $\mu_j(E, d') = p\mu_j(E, d) + (1 - p)\mu_j(E - d_i, c)$. But, by Proposition 6.4 and Corollary 6.2, we have $\mu_j(E - d_i, c) \leq \mu_j(E - d_i, d_{-i}) \leq \mu_j(E, d)$. Therefore, $\mu_j(E, d') = p\mu_j(E, d) + (1 - p)\mu_j(E - d_i, c) \leq p\mu_j(E, d) + (1 - p)\mu_j(E, d) = \mu_j(E, d)$. □

Trivially, if a rule $\mathcal{R}$ satisfies other-regarding claim monotonicity then it satisfies claim monotonicity. So, from Proposition 6.5 we have that the core-center rule satisfies this property. Nevertheless, we choose to give an alternative proof of this fact in Proposition 4.5, by means of a decomposition of the core, that analyzes the modifications of the core structure implied by a change of one single claim. Note, also, that claim monotonicity can be establish by showing that $\frac{\partial \mu_i}{\partial d_i} \geq 0$ for all $i \in N$. These derivatives can be computed using Lasserre's result (see Theorem A.3).

Our next goal is to show that the core-center rule satisfies $\frac{1}{|N|}$-truncated-claims lower bounds on awards, that is, the core-center rule guarantees a minimal share to every agent equal to one nth her claim truncated at the amount to be divided. The core-center rule also satisfies $\frac{1}{|N|}$-min-of-claim-and-deficit lower bounds on losses, since this property and $\frac{1}{|N|}$-truncated-claims lower bounds on awards are dual properties.

Proposition 6.6. *The core-center rule satisfies $\frac{1}{|N|}$-truncated-claims lower bounds on awards and $\frac{1}{|N|}$-min-of-claim-and-deficit lower bounds on losses.*

Proof. Let $(E, d) \in B^N$ be a bankruptcy problem such that $0 \leq d_1 \leq \cdots \leq d_n$. If $E \in [0, d_1]$ then, according to Proposition B.1, $\mu_j(E, d) = \frac{E}{n} = \frac{1}{n} \min\{E, d_j\}$ for all $j \in N$. If $E \in [d_1, d_2]$ then by Lemma E.1, $\mu_j(E, d) \geq \frac{1}{n} \min\{E, d_j\}$ for all $j \in N$. Now, by repeatedly applying Lemma E.2 it is easy to see that $\mu_j(E, d) \geq \frac{1}{n} \min\{E, d_j\}$ for all $j \in N$ whenever $E \in [d_2, d_n]$. But if $E \geq d_n$ and $j \in N \backslash \{n\}$ then, by other-regarding claim monotonicity, $\mu_j(E, d) \geq \mu_j(E, (d_1, \ldots, d_{n-1}, E))$, and we already know that $\mu_j(E, (d_1, \ldots, d_{n-1}, E)) \geq \frac{1}{n} \min\{E, d_j\}$. Lastly, it is clear that $\mu_n(E, d) \geq \frac{E}{n} \geq \frac{d_n}{n} = \frac{1}{n} \min\{E, d_n\}$. □

Example 6.7. *Let $N = \{1, 2, 3\}$ and consider the vector of claims $d = (1, 3, 5) \in \mathbb{R}^N$. In this case $d(N) = 9$ and $d(N \backslash \{3\}) = 4 \leq \frac{1}{2} d(N) = 4.5 \leq d_3 = 5$. Therefore if $E \in [4, 5]$ then $\mu_1(E, d) = \frac{d_1}{2} = \frac{1}{2}$, $\mu_2(E, d) = \frac{d_2}{2} = \frac{3}{2}$ and $\mu_3(E, d) = E - 2$. The graphs of the coordinates $\mu_j(\cdot, d)$, $j = 1, 2, 3$, are depicted in Figure 6. As in Example 6.3, we can see that they are monotonically increasing and the symmetry with respect to $E = 4.5$ implied by the self-duality property. Observe also that $\mu_1(E, d) \leq \mu_2(E, d)$ for all $E \in [0, 9]$. In Figure 6*

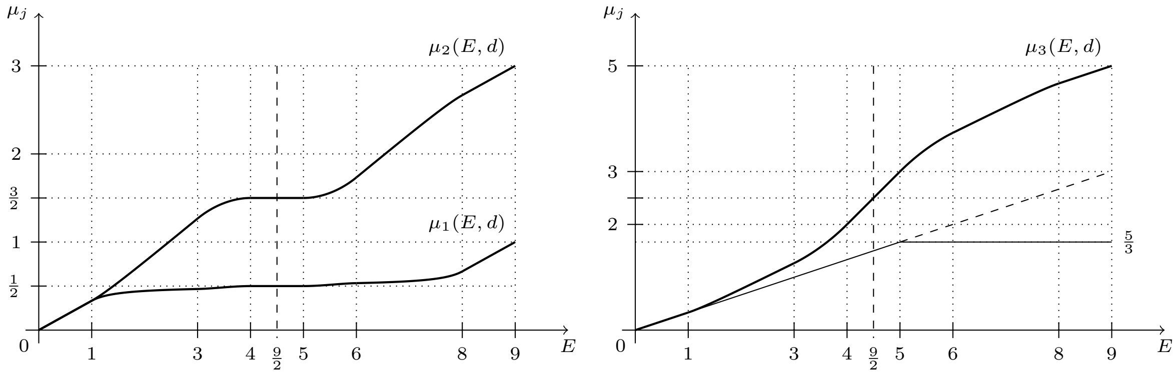

Figure 6: The coordinates of the core-center rule as functions of the endowment when $d = (1, 3, 5)$.

right, we compare $\mu_3(\cdot, d)$ with the corresponding piecewise linear function given by the $\frac{1}{|N|}$-truncated-claims lower bounds on awards, in fact, $\mu_3(E, d) \geq \frac{E}{3}$ for all $E \in [0, 9]$.

7 Concluding remarks

Following a game-theoretic approach, we have thoroughly studied the behavior of the core-center rule for the bankruptcy problem. The definition of the core-center rule is very intuitive, it is the average of all the award vectors that are bounded from below by the minimal rights and bounded from above by the truncated claims. There are several algorithms to compute (or estimate) the core-center of an arbitrary balanced game. For bankruptcy games, in Section 3 we have sketched a procedure to exactly calculate the core-center rule when there are only three claimants. Mirás Calvo et al. (2020a) extend the idea to an arbitrary population: the imputation set of a given bankruptcy game can be partitioned through cores of particular bankruptcy games. This decomposition provides a backwards recurrence algorithm to compute the core-center rule.

The continuity property guarantees that small changes in the claims vector or in the endowment lead to small changes in the recommendation made by the rule. Loosely speaking, a variation, no matter how small, in the initial endowment produces a change in the core of the associated bankruptcy game. The core-center rule is highly sensitive to such changes. In fact, we have proved that the core-center rule, for problems with at least three claimants, not only varies with continuity with respect to the endowment but also that the rate at which the rule changes varies also with continuity. Among the standard bankruptcy rules only the proportional rule is endowment differentiable.

The Lorenz order is the main criterion to rank rules. In order to compare a pair of awards vectors with the Lorenz ordering, first one has to rearrange the coordinates of each vector in a non-decreasing order. Then, we say that the former Lorenz-dominates the latter if all the cumulative sums of the rearranged coordinates are greater with the first vector than with the second. Certainly, the enumeration of properties satisfied by the core-center rule given in this paper is not exhaustive. Mirás Calvo et al. (2020b) add several properties to the list and use them to compare, using the Lorenz criterion, the core-center rule and the nine central bankruptcy rules considered by Bosmas and Lauwers (2011).

Some division rules that correspond to a solution to coalitional games can be characterized as the solution of an optimization problem. For instance, the Talmud rule corresponds to the nucleolus. Therefore, the Talmud rule minimizes, according to the lexicographical order, the vector of dissatisfactions, once decreasingly ordered its components, among all efficient payoff vectors, that is, minimizes the maximal complaint that a coalition might raise against a proposed division. Many important values for coalitional games are known to arise from least square optimization problems. Hokari (2000) shows that the constrained equal awards rule, that corresponds to the Dutta-Ray solution, is the one at which the variance of the amounts received by all of the claimants is minimized, that is,

$$CEA(E, d) = \arg\min_{x \in C(E,d)} \sum_{i=1}^{|N|} \left(x_i - \tfrac{E}{|N|}\right)^2.$$

Given a bankruptcy problem $(E, d) \in B^N$, the random arrival rule, i.e. the Shapley value of the corresponding bankruptcy game $v \in G^N$, is the least square value (see Ruíz et al. (1998)), given by

$$RA(E,d) = \underset{x \in H(E,d)}{\arg\min} \sum_{S \subsetneq N} \alpha(S)\big(v(S) - x(S)\big)^2,$$

where $\alpha(S) = (|S|-1)!(|N|-|S|-1)!$ for $S \subsetneq N$.

Now, if a rule $\mathcal{R}$ recommends an award a_i to claimant i, then, for each $x \in C(E,d)$, the value $(x_i - a_i)^2$ is the square deviation of x_i with respect to that claimant's award. So, $\int_{C(E,d)} (x_i - a_i)^2 dm$ is the sum (integral) over all stable allocations of the square deviations with respect to a_i. It follows from the definition of the core-center rule as the mean value of the uniform distribution over the core of the associated bankruptcy game, that the core-center rule is the one that, for each claimant, minimizes the integral over all stable allocations of the square deviations. Therefore,

$$\mu_i(E,d) = \underset{a_i \in I_i}{\arg\min} \int_{C(E,d)} (x_i - a_i)^2 dm,$$

where $I_i = [m_i(E,d), t_i(E,d)]$ for all $i \in N$.

We introduced a property called single-agent average consistency that help us characterized the core-center rule. Another axiomatic characterization could be worked out from the already mentioned decomposition of the imputation set of a given bankruptcy game by cores of particular bankruptcy games. In fact, González-Díaz and Sánchez-Rodríguez (2009) introduced a requirement, called the trade-off property, to characterize the core-center in the general class of balanced games. The basic idea of the characterization is to decompose the original core in pieces that are "simple" cores of games. The solution in these "simple" cores is described by standard axioms (efficiency and symmetry properties). Then, the trade-off property is used to obtain the core-center as the weighted sum of the solution applied to the "simple" cores of the decomposition. The downside when considering bankruptcy problems is that the pieces of the core dissection are not cores of bankruptcy games themselves but translates of cores of bankruptcy games. Therefore, one needs to enlarge the class of problems.

An interesting topic is to implement cooperative solutions through non-cooperative procedures. In that respect, Tsay and Yeh (2019) provide strategic implementations of the constrained equal awards rule, the constrained equal losses rule, the proportional rule, and the Talmud rule. An open question for future research is to find a strategic procedure of the core-center rule.

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Appendix

A The volume of the core of a bankruptcy game

Let $v \in G^N$. The imputation set $I(v)$ is nonempty if and only if $\Delta = v(N) - \sum_{k\in N} v(k) \geq 0$. In that case, $I(v)$ is the regular simplex, contained in the hyperplane $H(v) \subset \mathbb{R}^N$, spanned by the points $a^i = (a^i_1, \dots, a^i_n) \in \mathbb{R}^N$, $i \in N$, where $a^i_j = \begin{cases} v(j) & \text{if } j \in N\backslash\{i\} \\ v(N) - \sum_{k\neq i} v(k) & \text{if } j = i \end{cases}$. Then $\sqrt{2}\Delta$ is the common edge length. The center of gravity of $I(v)$ is the arithmetic mean of its extreme points, $\mu(I(v)) = \sum_{i=1}^{n} \frac{a^i}{n}$, so $\mu_i(I(v)) = v(i) + \frac{\Delta}{n}$ for all $i \in N$. The $(n-1)$-volume of $I(v)$ is $\mathrm{Vol}(I(v)) = \frac{\sqrt{n}}{(n-1)!}\Delta^{n-1}$ (see, for instance, Gritzman and Klee (1994)).

It is a well known fact that if $(E, d) \in B^N$ is a bankruptcy problem and $v \in G^N$ is the associated bankruptcy game then v is a convex game. Therefore, $E = v(N) \geq \sum_{k\in N} v(k)$, so $I(E, d) \neq \emptyset$. Let $d = (d_1, \dots, d_n) \in \mathbb{R}^N$ be a sorted vector of claims in ascending order, i.e., $0 < d_1 \leq \cdots \leq d_n$. Clearly $(E, d) \in B^N$ for each $E \in [0, d(N)]$ so we can define the volume function $V(\cdot, d)\colon [0, d(N)] \to \mathbb{R}$ as $V(E, d) = \mathrm{Vol}_{n-1}(C(E, d))$. Our aim is to thoroughly analyze the volume function.

For a bankruptcy problem $(E, d) \in B^N$ with two claimants, $N = \{1, 2\}$, and $d = (d_1, d_2) \in \mathbb{R}^N$ such that $0 \leq d_1 \leq d_2$, it is clear that $C(E, d) = I(E, d)$ is the line segment with endpoints $\big(m_1(E, d), E - m_1(E, d)\big)$ and $\big(E - m_2(E, d), m_2(E, d)\big)$. The core of a two claimants bankruptcy problems is shown in Figure 3. Then

$$V(E, d) = \sqrt{2}\big(E - v(1) - v(2)\big) = \begin{cases} \sqrt{2}E & \text{if } 0 \leq E \leq d_1 \\ \sqrt{2}d_1 & \text{if } d_1 \leq E \leq d_2 \\ \sqrt{2}(d_1 + d_2 - E) & \text{if } d_2 \leq E \leq d_1 + d_2 \end{cases}.$$

The graph of the piecewise linear function $V(\cdot, d)$ is depicted in Figure 7.

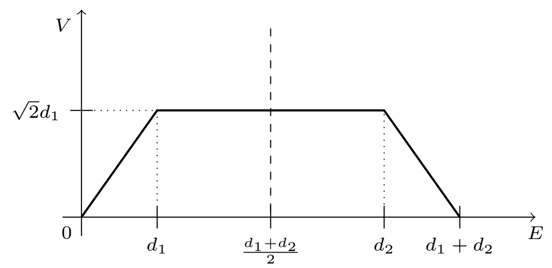


Figure 7: The volume function when $|N| = 2$.

The following result identifies the bankruptcy problems for which the core is not full dimensional when there are at least three claimants.

Lemma A.1. *Let $(E, d) \in B^N$ be a bankruptcy problem with $|N| \geq 3$.*

1. *$C(E, d) = I(E, d)$ if and only if either $E \leq d_i$ for all $i \in N$ or $E = d(N)$.*
2. *$V(E, d) = 0$ if and only if one of the following conditions holds:*

   1. $E = 0$. *2. $E = d(N)$.* *3. There is $i \in N$ with $d_i = 0$.*

Proof. If $|N| \geq 3$, it holds that $C(E, d) = I(E, d)$ if and only if either $\min\{E, d_i\} = E$ or $v(i) = \min\{E, d_i\}$ for all $i \in N$, or equivalently, if either $E \leq d_i$ for all $i \in N$ or $E = d(N)$. Naturally, when $E = 0$ or $E = d(N)$ the core is a singleton, in fact, $C(0, d) = \{(0, \dots, 0)\}$ and $C(d(N), d) = \{d\}$. So, in addition to these two cases, $V(E, d) = 0$ if and only if there is at least a player $i \in N$ for which $\max\{0, E - d(N\backslash\{i\})\} = \min\{E, d_i\}$. $\square$

If $V(E,d) > 0$ then $C(E,d)$ is a $(n-1)$-dimensional manifold contained in $H(E,d)$, the efficiency hyperplane. Therefore $H(E,d)$ is the tangent space at each point of the manifold. The vector $(1,1,\ldots,1) \in \mathbb{R}^N$ is normal to the manifold at each point and it has length $\sqrt{n}$. For each $i \in N$ the transformation $\pi_i \colon \mathbb{R}^{N\setminus\{i\}} \to \mathbb{R}^N$, $\pi_i(x_{-i}) = \big(x_{-i}, E - x_{-i}(N\setminus\{i\})\big) \in \mathbb{R}^N$ defines a coordinate system for $C(E,d)$, so that $C_i(E,d) = \pi_i^{-1}(C(E,d)) \subset \mathbb{R}^{N\setminus\{i\}}$ is the projection of the core onto $\mathbb{R}^{N\setminus\{i\}}$ that simply "drops" the ith-coordinate, see Figure 8. In fact,

$$C_i(E,d) = \Big\{y \in \mathbb{R}^{N\setminus\{i\}} \colon r_i(E,d) \le y(N\setminus\{i\}) \le R_i(E,d),\ m_j(E,d) \le y_j \le t_j(E,d) \text{ for all } j \in N\setminus\{i\}\Big\}.$$

Let $V_i(E,d) = \mathrm{Vol}_{n-1}(C_i(E,d))$ be the volume of the projection of the core $C(E,d)$ onto $\mathbb{R}^{N\setminus\{i\}}$.

Next, we compute the volume function in some easy cases. It turns out that $V(\cdot,d)$ is a symmetric function with respect to the half-sum of the claims. In particular, we show that when $d(N\setminus\{n\}) \le E \le d_n$, the nth-projection of the core, $C_n(E,d)$, is a $(n-1)$-rectangle that does not depend on the endowment E, and so $V(\cdot,d)$ is constant on $[d(N\setminus\{n\}), d_n]$. Moreover, the volume of the core is $\sqrt{n}$ times the volume of its ith-projection onto $\mathbb{R}^{N\setminus\{i\}}$.

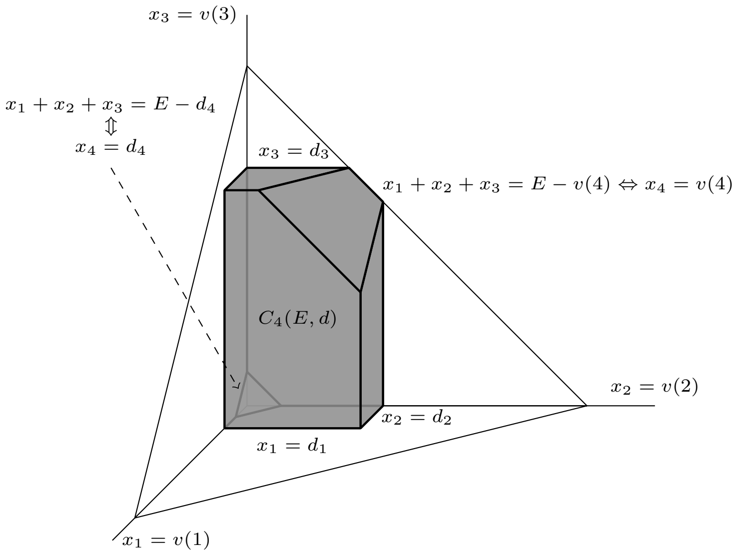


Figure 8: The projection $C_4(E,d)$.

Lemma A.2. *Let $(E,d) \in B^N$ be a bankruptcy problem with $|N| \ge 3$.*

1. *If $E \in [0, d_1]$ then $V(E,d) = \frac{\sqrt{n}}{(n-1)!}E^{n-1}$.*
2. *If $E \in [\frac{1}{2}d(N), d(N)]$ then $d(N) - E \in [0, \frac{1}{2}d(N)]$ and $V(E,d) = V(d(N)-E, d)$.*
3. *$V(E,d) = \sqrt{n}V_i(E,d)$ for all $i \in N$.*
4. *If $E \in [d(N\setminus\{n\}), d_n]$ then $C_n(E,d) = [0,d_1] \times \cdots \times [0, d_{n-1}]$ and $V(E,d) = \sqrt{n}\prod_{j=1}^{n-1} d_j$.*

Proof. We know that if $0 \le E \le d_1$ then $C(E,d) = I(E,d)$. The second statement follows directly from Proposition 3.2, the self-duality of the core. Finally, let m be the $(n-1)$-dimensional Lebesgue measure. Then, for all $i \in N$,

$$V(E,d) = \int_{C(E,d)} dm = \int_{\pi_i^{-1}(C(E,d))} \sqrt{n}\ dm = \sqrt{n}V_i(E,d).$$

If $E \in [d(N\setminus\{n\}), d_n]$ then $r_n(E,d) = 0$ and $R_n(E,d) = d(N\setminus\{n\})$. Also $m_j(E,d) = 0$ and $t_j(E,d) = d_j$ for all $j \in N\setminus\{n\}$. Therefore $C_n(E,d) = \big\{y \in \mathbb{R}^{N\setminus\{n\}} : 0 \le y_j \le d_j, \text{ for } j \in N\setminus\{n\}\big\} = [0,d_1]\times\cdots\times[0,d_{n-1}]$. □

When $|N| = 2$, the volume function is not differentiable at d_1 and d_2 (see Figure 7). On the contrary, if $|N| \geq 3$ then $V(\cdot, d)$ is a differentiable function. In order to prove this property we rely on a general result relating the volume of a convex polyhedron with the volume of its faces. Let $K(b) = \{x \in \mathbb{R}^n : Ax \leq b\}$ be a convex polyhedron, where A is an $m \times n$ matrix and b is a m-vector, and denote $V(b) = \mathrm{Vol}_n(K(b))$ its volume as a function of b. In general, V is a continuous function of b. Let $K_i(b) = \{x \in K(b) : a_i \cdot x = b_i\}$, $i = 1, \ldots, m$, be the ith-face of $K(b)$, where a_i is the ith-row of A and $a_i \cdot x$ is the scalar product in $\mathbb{R}^n$; and denote its $(n-1)$-volume by $V(b, i) = \mathrm{Vol}_{n-1}(K_i(b))$. If $V(b) > 0$ and $V(b, i) = 0$ for some $i = 1, \ldots, m$ then constraint $a_i \cdot x \leq b_i$ is redundant.

Theorem A.3 (Lasserre (1983))**.** *If $V(b) > 0$ and $V(b, i) > 0$ for all $i = 1, \ldots, m$, then V is differentiable at b and $\frac{\partial V}{\partial b_i}(b) = \frac{1}{||a_i||} V(b, i)$. Moreover, $V(b) = \frac{1}{n} \sum_{i=1}^{m} \frac{b_i}{||a_i||} V(b, i)$.*

Applying Theorem A.3 to the ith-projection $C_i(E, d)$ we show that the volume function is differentiable. Moreover, we can write the derivative of $V(\cdot, d)$ in terms of the volumes of the faces of $C(E, d)$.

Theorem A.4. *Let $d = (d_1, \ldots, d_n) \in \mathbb{R}^N$ be a vector of claims such that $0 < d_1 \leq \cdots \leq d_n$. If $|N| \geq 3$ then $V(\cdot, d) \colon [0, d(N)] \to \mathbb{R}$ is a continuously differentiable function. Moreover,*

1. *if $E \in [\frac{1}{2} d(N), d(N)]$ then $\frac{\partial V}{\partial E}(E, d) = -\frac{\partial V}{\partial E}(d(N) - E, d)$.*
2. *$\frac{\partial V}{\partial E}(E, d) = 0$ if $E \in [d(N \backslash \{n\}), d_n]$.*
3. *if $E \in [0, \frac{1}{2} d(N)]$ then, for all $i \in N$,*

$$\begin{aligned} \frac{\partial V}{\partial E}(E, d) &= \frac{\sqrt{n}}{\sqrt{n-1}} \big(V(E, d_{-i}) - V(r_i(E, d), d_{-i}) \big) \\ &= \begin{cases} \frac{\sqrt{n}}{\sqrt{n-1}} V(E, d_{-i}) & \text{if } 0 \leq E \leq d_i \\ \frac{\sqrt{n}}{\sqrt{n-1}} \big(V(E, d_{-i}) - V(E - d_i, d_{-i}) \big) & \text{if } d_i < E \leq \frac{1}{2} d(N) \end{cases}. \end{aligned}$$

Proof. Observe that, from Proposition 3.2, it suffices to prove that $V(\cdot, d)$ is differentiable on $[0, \frac{1}{2} d(N)]$. If that is the case, the first statement follows directly. Also, from Lemma A.2, we know that if $E \in [d(N \backslash \{n\}), d_n]$ then $V(E, d) = \sqrt{n} \prod_{i=1}^{n-1} d_i$ and, therefore, $\frac{\partial V}{\partial E}(E, d) = 0$. Since $\frac{1}{2} d(N)$ is halfway between $d(N \backslash \{n\})$ and d_n we can assume that $E \leq \min\{\frac{1}{2} d(N), d(N \backslash \{n\})\}$ in which case $E \leq d(N \backslash \{n\}) \leq \cdots \leq d(N \backslash \{1\})$ and $R_i(E, d) = E$ for all $i \in N$. Now, fix $i \in N$. Clearly,

$$C_i(E, d) = \big\{ y \in \mathbb{R}^{N \backslash \{i\}} \colon r_i(E, d) \leq y(N \backslash \{i\}) \leq E,\ 0 \leq y_j \leq t_j(E, d) \text{ for all } j \in N \backslash \{i\} \big\}.$$

If $\min\{E, d_j\} = E$ then the constraint $y_j \leq E$ is redundant because $y_j \leq y(N \backslash \{i\}) \leq E$. On the other hand, $r_i(E, d) = E - d_i$ only if $E \geq d_i$. Observe that, in order to apply Theorem A.3, the only constraints that depend on E are $y(N \backslash \{i\}) \leq E$ and, if $E \geq d_i$, $E - d_i \leq y(N \backslash \{i\})$. Note that the last constraint must be written as $-y(N \backslash \{i\}) \leq d_i - E$, so the corresponding derivative with respect to E will be negative. But,

$$C_i(E, d) \cap \big\{ y \in \mathbb{R}^{N \backslash \{i\}} \colon y(N \backslash \{i\}) = E \big\} = C(E, d_{-i})$$

and, if $E \geq d_i$,

$$C_i(E, d) \cap \big\{ y \in \mathbb{R}^{N \backslash \{i\}} \colon y(N \backslash \{i\}) = E - d_i \big\} = C(E - d_i, d_{-i}).$$

Combining Theorem A.3 and the chain rule we conclude that $V(\cdot, d)$ is differentiable at E and

$$\frac{\partial V}{\partial E}(E, d) = \sqrt{n} \frac{\partial V_i}{\partial E}(E, d) = \frac{\sqrt{n}}{\sqrt{n-1}} \big(V(E, d_{-i}) - V(r_i(E, d), d_{-i}) \big).$$

Therefore $V(\cdot, d)$ is a differentiable function on $[0, \frac{1}{2} d(N)]$ except, perhaps, at the points $0, d_1, \ldots, d_n$ and $d(N \backslash \{n\})$, and its derivatives are given by the expressions written above. Now, it is easy to check that $V(\cdot, d)$ is also differentiable at those points. □

The volume function $V(\cdot, d)$ is monotonically increasing on $[0, \frac{1}{2} d(N)]$ and monotonically decreasing on $[\frac{1}{2} d(N), d(N)]$. The graph of volume function $V(\cdot, d)$, depending on whether $d(N \backslash \{n\}) < d_n$ or not, is depicted in Figure 9.

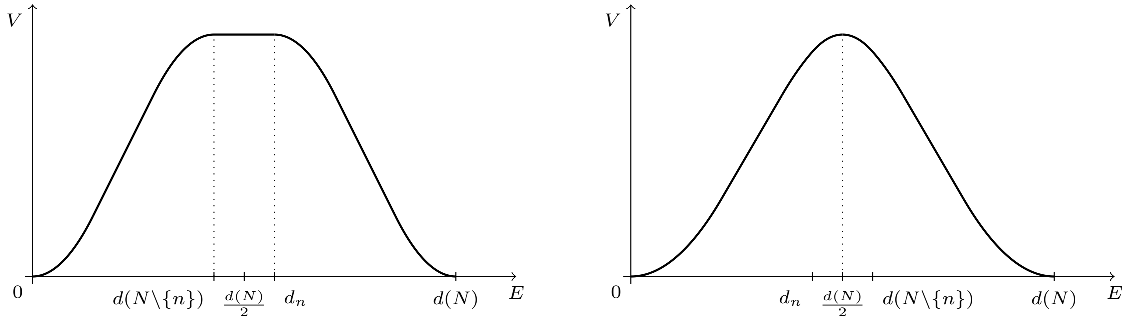


Figure 9: The volume of the core as a function of the endowment.

Proposition A.5. *Let* $d = (d_1, \ldots, d_n) \in \mathbb{R}^N$ *with* $0 < d_1 \leq \cdots \leq d_n$.

1. *If* $d(N\backslash\{n\}) > d_n$ *then* $V(\cdot, d)$ *is strictly increasing on* $[0, \frac{1}{2}d(N)]$ *and* $V(\cdot, d)$ *is strictly decreasing on* $[\frac{1}{2}d(N), d(N)]$, *so* $V(\cdot, d)$ *attains its maximum at* $E = \frac{1}{2}d(N)$.
2. *If* $d(N\backslash\{n\}) \leq d_n$ *then* $V(\cdot, d)$ *is strictly increasing on* $[0, d(N\backslash\{n\})]$, *it is strictly decreasing on* $[d_n, d(N)]$ *and it is constant on* $[d(N\backslash\{n\}), d_n]$.

Proof. The result when $|N| = 2$ is clear. Let $|N| \geq 3$. We know from Lemma A.2 that $V(\cdot, d)$ is constant on $[d(N\backslash\{n\}), d_n]$ whenever $d(N\backslash\{n\}) \leq d_n$, so it suffices to prove that $V(\cdot, d)$ is strictly increasing on $[0, \min\{\frac{1}{2}d(N), d(N\backslash\{n\})\}]$. Since $V(\cdot, d)$ is a differentiable function, we have to prove that $\frac{\partial V}{\partial E}(E, d) > 0$ for all $E < \min\{\frac{1}{2}d(N), d(N\backslash\{n\})\}$. We proceed by induction on n. Assume that $V(\cdot, d_{-n})$ is strictly increasing on $\big[0, \min\big\{\frac{1}{2}d(N\backslash\{n\}), d\big(N\backslash\{n, n-1\}\big)\big\}\big]$. Now, from Theorem A.4, $\frac{\partial V}{\partial E}(E, d) = \frac{\sqrt{n}}{\sqrt{n-1}}V(E, d_{-n}) > 0$ if $0 < E < d_n$. On the other hand, if $E \in [d_n, \frac{1}{2}d(N))$ and $\frac{1}{2}d(N) \leq d(N\backslash\{n\})$ then, again from Theorem A.4, $\frac{\partial V}{\partial E}(E, d) = \frac{\sqrt{n}}{\sqrt{n-1}}\big(V(E, d_{-n}) - V(E - d_n, d_{-n})\big)$. But $E - d_n \leq \frac{1}{2}\big(d(N\backslash\{n\}) - d_n\big) \leq \frac{1}{2}d(N\backslash\{n\})$ and $E \leq d(N\backslash\{n\}) \leq d\big(N\backslash\{n, n-1\}\big) + d_n$, so $E - d_n \leq d\big(N\backslash\{n, n-1\}\big)$. Therefore $E - d_n \leq \min\big\{\frac{1}{2}d(N\backslash\{n\}), d\big(N\backslash\{n, n-1\}\big)\big\}$. Now, $\frac{1}{2}d(N\backslash\{n\})$ is halfway between $E - d_n$ and $d(N) - E$, so by the induction hypothesis and the symmetry of $V(\cdot, d_{-n})$, we have that $V(E - d_n, d_{-n}) \leq V(u, d_{-n})$ for all $u \in [E - d_n, d(N) - E]$. In particular, since $E - d_n < E < d(N) - E$ then $V(E - d_n, d_{-n}) < V(E, d_{-n})$ and, consequently, $\frac{\partial V}{\partial E}(E, d) > 0$. □

B Computation of the core-center rule

In some particular cases the computation of the core-center rule can be carried out easily or can be greatly simplified.

Proposition B.1. *Let* $(E, d) \in B^N$ *be a bankruptcy problem such that* $|N| \geq 3$ *and* $0 \leq d_1 \leq \cdots \leq d_n$.

1. *If* $E \leq d_1$ *then* $\mu_i(E, d) = \frac{E}{n}$ *for all* $i \in N$.
2. *If* $d_i \geq E$ *for some* $i \in N$, *then* $\mu_i(E, d) = \mu_j(E, d)$ *for all* $j > i$.
3. *If* $E \geq d(N\backslash\{1\})$ *then* $\mu_i(E, d) = d_i - \frac{d(N)-E}{n}$ *for all* $i \in N$.
4. *If* $E = \frac{1}{2}d(N)$ *then* $\mu(E, d) = \frac{d}{2}$.
5. *If* $d(N\backslash\{n\}) \leq E \leq d_n$ *then* $\mu_j(E, d) = \frac{d_j}{2}$ *for all* $j \in N\backslash\{n\}$ *and* $\mu_n(E, d) = E - \frac{1}{2}d(N\backslash\{n\})$.
6. *If* $E > d(N\backslash\{i\})$ *for some* $i \in N$ *then* $\mu_j(E, d) = (d_j - d_i) + \mu_i(E, d)$ *for all* $j > i$.

Proof. The first statement follows directly from Lemma A.1 while the second is a consequence of equal treatment of equals and claims truncation invariance (see Proposition 4.1). The third and fourth statements hold because μ satisfies self-duality. The fifth comes from Lemma A.2 since $\mu_j(E, d) = \mu_j(C_n(E, d))$ for all $j \in N\backslash\{n\}$ and $\mu(E, d) \in H(E, d)$. Finally, if $E > d(N\backslash\{i\})$ for some $i \in N$ then $d(N) - E < d_i$ and $\mu_i(d(N) - E, d) = \mu_j(d(N) - E, d)$ for all $j > i$. Now, by self-duality, $\mu_i(d(N) - E, d) = d_i - \mu_i(E, d)$ and $\mu_j(d(N) - E, d) = d_j - \mu_j(E, d)$, so $\mu_j(E, d) = (d_j - d_i) + \mu_i(E, d)$. □

Next, we establish the fundamental relationship between the core-center rule and the center of gravity of the ith-projection $C_i(E,d)$.

Proposition B.2. *Let $(E,d) \in B^N$ be a bankruptcy problem and $i \in N$. If $|N| \geq 3$ then $\mu_j(E,d) = \mu_j(C_i(E,d))$ for all $j \in N\backslash\{i\}$ and $\mu_i(E,d) = E - \sum_{j \in N\backslash\{i\}} \mu_j(E,d)$.*

Proof. Let m be the $(n-1)$-dimensional Lebesgue measure. We know from Lemma A.2 that $V(E,d) = \sqrt{n}V_i(E,d)$. Then $\mu_j(E,d) = \frac{1}{V(E,d)} \int_{C(E,d)} x_j dm = \frac{1}{V(E,d)} \int_{C_i(E,d)} \sqrt{n} x_j dm = \frac{1}{V_i(E,d)} \int_{C_i(E,d)} x_j dm = \mu_j(C_i(E,d))$ for all $j \in N\backslash\{i\}$. Finally, the expression for $\mu_i(E,d)$ follows because $\sum_{k \in N} \mu_k(E,d) = E$. $\square$

The properties already stated are very helpful to simplify the computation of the core-center rule in some cases, in particular when the number of claimants is small. The next example illustrates how to apply Proposition B.2 to the computation of the core-center rule when $|N| = 3$.

Example B.3. *Let $N = \{1,2,3\}$, $d = (1,2,2) \in \mathbb{R}^N$ and $E = 3$. Now $d(N) = d_1 + d_2 + d_3 = 5$ and*

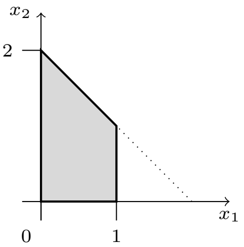


Figure 10: The projection $C_3(d(N)-E,d)$.

$E > \frac{1}{2}d(N)$. Since μ satisfies self-duality, $\mu(E,d) = d - \mu(d(N)-E,d)$. But $d(N) - E = 2$ so

$$C_3(d(N)-E,d) = \big\{(x_1,x_2) \in \mathbb{R}^2 \colon 0 \leq x_1 \leq 1,\ 0 \leq x_2,\ x_1 + x_2 \leq 2\big\}.$$

Now, $V_3(d(N)-E,d) = \int_0^1 \int_0^{2-x_1} dx_2 dx_1 = \frac{3}{2}$. By Proposition B.2,

$$\mu_1(d(N)-E,d) = \mu_1(C_3(d(N)-E,d)) = \frac{2}{3}\Big(\int_0^1 \int_0^{2-x_1} x_1 dx_2 dx_1\Big) = \tfrac{4}{9}.$$

Applying equal treatment of equals we have that $\mu_2(d(N)-E,d) = \mu_3(d(N)-E,d) = \frac{7}{9}$. So, finally $\mu(E,d) = d - \mu(d(N)-E,d) = (\frac{5}{9}, \frac{11}{9}, \frac{11}{9})$.

C Decompositions of the core of a bankruptcy game

Let $(E,d) \in B^N$. We give two different decompositions of the core of the bankruptcy game $C(E,d)$ as the union of two sets $C(E,d) = C_1 \cup C_2$, with $\mathrm{Vol}(C_1 \cap C_2) = 0$, and such that C_1 and C_2 are cores of bankruptcy games, possibly translated by a specific vector, associated with bankruptcy problems obtained from (E,d). For each $i \in N$, denote $e^i \in \mathbb{R}^N$ the vector with 1 in the ith-coordinate and 0's elsewhere.

Assume that $(E,d) \in B^N$ is a bankruptcy problem for which $m(E,d) = 0$. Let $i \in N\backslash\{n\}$ such that $d_i < E$. Intuitively, the first decomposition translates order preservation in awards in terms of cores of bankruptcy games. In fact, it shows that $C(E,d)$ contains the core of the bankruptcy game associated with the problem where agent $i+1$ claims d_i, that is $C_1 = C(E,a)$ with $a = (d_1, \ldots, d_i, d_i, d_{i+2}, \ldots, d_n)$. The other stable allocations, except for a set of null measure, belong to the difference $C_2 = C(E,d)\backslash C_1$. Basically, C_2 is the the core of the bankruptcy game associated with the bankruptcy problem with endowment $E - d_i$ and where agent $i+1$ claims the difference $d_{i+1} - d_i$, but with agent $i+1$ receiving and extra award d_i.

Proposition C.1. *Let $(E, d) \in B^N$ be a bankruptcy problem such that $m(E, d) = 0$. Let $i \in N \backslash \{n\}$ such that $d_i < E$ and denote $a = (d_{-(i+1)}, d_i)$, $b = d_i e^{i+1}$, and $c = d - b$. Then,*

$$C(E, d) = C(E, a) \cup \big(b + C(E - d_i, c)\big)$$

and $\text{Vol}\Big(C(E, a) \cap \big(b + C(E - d_i, c)\big)\Big) = 0.$

Proof. Since $m(E, d) = 0$, the bankruptcy game $v \in G^N$ associated with the bankruptcy proble $(E, d) \in B^N$ is a zero-normalized game. Fix $i \in N \backslash \{n\}$ such that $d_i < E$. First, note that (E, a) and $(E - d_i, c)$ are, in fact, bankruptcy problems, because $v(i + 1) = 0$. Then, let $v_a, v_c \in G^N$ be the bankruptcy games associated with $(E, a) \in B^N$ and $(E - d_i, c) \in B^N$ respectively. We have $v_a(k) = 0$ if $k \le i + 1$ and $v_c(k) = 0$ for all $k \in N$. Moreover, $v_a(k) \le \min\{E - d_i, d_k\}$ if $k > i + 1$. Then,

$$\begin{aligned} C(E, d) &= \big\{x \in \mathbb{R}^N : x(N) = E,\ 0 \le x_k \le d_k \text{ if } k \le i,\ 0 \le x_k \le \min\{E, d_k\} \text{ if } k > i\big\} \\ C(E, a) &= \left\{y \in \mathbb{R}^N : y(N) = E,\ 0 \le y_{i+1} \le d_i,\ \begin{array}{l} 0 \le y_k \le d_k \text{ if } k \le i \\ v_a(k) \le y_k \le \min\{E, d_k\} \text{ if } k > i + 1 \end{array}\right\} \\ C(E - d_i, c) &= \left\{z \in \mathbb{R}^N : z(N) = E - d_i,\ \begin{array}{l} 0 \le z_k \le \min\{E - d_i, d_k\} \text{ if } k \ne i + 1 \\ 0 \le z_{i+1} \le \min\{E - d_i, d_{i+1} - d_i\} \end{array}\right\}. \end{aligned}$$

It is easy to check that $C(E, a) \subset C(E, d)$, $b + C(E - d_i, c) \subset C(E, d)$, and that $C(E, a)$ and $b + C(E - d_i, c)$ are separated by the hyperplane $x_{i+1} = d_i$. But if $x \in C(E, d)$ and $x \notin C(E, a)$ then either $d_i \le x_{i+1} < \min\{E, d_{i+1}\}$ in which case $0 \le x_{i+1} - d_i < \min\{E - d_i, d_{i+1} - d_i\}$; or $0 \le x_k < v_a(k)$ for some $k > i + 1$ in which case $0 \le x_k < \min\{E - d_i, d_k\}$. Then $z = x - b \in C(E - d_i, c)$. □

The second decomposition reflects how claim monotonicity can be interpreted in terms of cores of bankruptcy games. Let $(E, d) \in B^N$ and assume that agent's i claim increases from d_i to d'_i and consider the bankruptcy problem $(E, d') \in B^N$, with $d' = (d_1, \ldots, d_{i-1}, d'_i, d_{i+1}, \ldots, d_n)$. Let $(E - d_i, c) \in B^N$, with $c = (d_1, \ldots, d_{i-1}, d'_i - d_i, d_{i+1}, \ldots, d_n)$, be the bankruptcy problem obtained from (E, d) by reducing the endowment by d_i and where agent i claims the difference $d'_i - d_i$. We show that $C(E, d) \subset C(E, d')$. In fact, any stable allocation that belongs to the core $C(E, d')$ is the sum of three allocations: one in the core $C(E, d)$, another in $C(E - d_i, c)$ and a fixed allocation that awards d_i to agent i.

Proposition C.2. *Let $(E, d) \in B^N$ be a bankruptcy problem and $i \in N \backslash \{n\}$ such that $d_i < E$. Let $d_i < d'_i \le d_{i+1}$ and denote $d' = (d_{-i}, d'_i)$, $b = d_i e^i$, and $c = d' - b$. Then,*

$$C(E, d') = C(E, d) \cup \big(b + C(E - d_i, c)\big)$$

and $\text{Vol}\Big(C(E, d) \cap \big(b + C(E - d_i, c)\big)\Big) = 0.$

Proof. Let $v \in G^N$ the bankruptcy game associated with the bankruptcy problem $(E, d) \in B^N$ and fix $i \in N \backslash \{n\}$ such that $d_i < E$. Certainly, (E, d') and $(E - d_i, c)$ are bankruptcy problems so denote $v_{d'}, v_c \in G^N$ the bankruptcy games associated with $(E, d') \in B^N$ and $(E - d_i, c) \in B^N$ respectively. We have $v_{d'}(i) = v(i)$, $v_c(i) = 0$, $v_c(j) = v_{d'}(j) \ge v(i)$ for $j \ne i$. Therefore,

$$\begin{aligned} C(E, d') &= \big\{x \in \mathbb{R}^N : x(N) = E,\ v(i) \le x_i \le \min\{E, d'_i\},\ v_{d'}(j) \le x_j \le \min\{E, d_j\} \text{ if } j \ne i\big\} \\ C(E, d) &= \big\{y \in \mathbb{R}^N : y(N) = E,\ v(i) \le y_i \le d_i,\ v(j) \le y_j \le \min\{E, d_j\} \text{ if } j \ne i\big\} \\ C(E - d_i, c) &= \left\{z \in \mathbb{R}^N : z(N) = E - d_i,\ \begin{array}{l} 0 \le z_i \le \min\{E - d_i, d'_i - d_i\} \\ v_{d'}(j) \le z_j \le \min\{E - d_i, d_j\} \text{ if } j \ne i \end{array}\right\}. \end{aligned}$$

It is easy to check that $C(E, d) \subset C(E, d')$, $b + C(E - d_i, c) \subset C(E, d')$, and that $C(E, d)$ and $b + C(E - d_i, c)$ are separated by the hyperplane $x_i = d_i$. But if $x \in C(E, d')$ and $x \notin C(E, d)$ then either $d_i < x_i \le \min\{E, d'_i\}$ in which case $0 < x_i - d_i \le \min\{E - d_i, d'_i - d_i\}$, or $v_{d'}(j) \le x_j < v(j)$ for some $j \ne i$, in which case $v_{d'}(j) \le x_j < \min\{E - d_i, d_j\}$ because $v(j) \le \min\{E - d_i, d_j\}$. Then $z = x - b \in C(E - d_i, c)$. □

D Integral representations of the core-center rule

For each $i \in N$ consider the function $g_i\colon (0,d(N)) \times [0,d(N\backslash\{i\})] \to \mathbb{R}$ defined as:

$$g_i(E,u) = \frac{\sqrt{n}}{\sqrt{n-1}}\frac{V(u,d_{-i})}{V(E,d)}, \text{ for all } (E,u) \in (0,d(N)) \times [0,d(N\backslash\{i\})].$$

Clearly, $g_i(E,u) \geq 0$ for all $(E,u) \in (0,d(N)) \times [0,d(N\backslash\{i\})]$. According to Theorem A.4, g_i is a continuous differentiable function, $\frac{\partial g_i}{\partial E}$ is continuous, and $\frac{\partial g_i}{\partial E}(E,u) = -\frac{\sqrt{n}}{\sqrt{n-1}}\frac{V(u,d_{-i})}{(V(E,d))^2}\frac{\partial V}{\partial E}(E,d)$.

Theorem D.1. *Assume that* $|N| \geq 3$. *If* $(E,d) \in B^N$ *and* $i \in N$ *then* $\int_{v(i)}^{\min\{E,d_i\}} g_i(E,E-s)ds = 1$ *and*

$$\mu_j(E,d) = \int_{v(i)}^{\min\{E,d_i\}} \mu_j(E-s,d_{-i})g_i(E,E-s)ds \text{ for all } j \in N\backslash\{i\}.$$

Proof. Let us simplify the notation by writing $a_i = v(i)$, $b_i = \min\{E,d_i\}$, $I_i = [a_i,b_i]$, $C = C(E,d)$ and $C_{x_i} = C(E-x_i,d_{-i})$. The transformation $\pi_n(x_1,\ldots,x_{n-1}) = (x_1,\ldots,x_{n-1},E-x_1-\cdots-x_{n-1})$ defines a parametrization of the hyperplane $x_1+\cdots+x_n = E$. The vector $(1,1,\ldots,1) \in \mathbb{R}^n$ is normal to the hyperplane at each point and it has length $\sqrt{n}$. For each $x_i \in I_i$ the transformation $h_{x_i}(x_1,\ldots,x_{i-1},x_{i+1},\ldots,x_{n-1},(E-x_i)-x_1-\cdots-x_{i-1}-x_{i+1}-\cdots-x_{n-1})$ defines a parametrization[1] of the hyperplane $x_1+\cdots+x_{i-1}+x_{i+1}+\cdots+x_{n-1} = E-x_i$. The vector $(1,1,\ldots,1) \in \mathbb{R}^{n-1}$ is normal to the hyperplane at each point and it has length $\sqrt{n-1}$. From Proposition 3.1 it is easy to check that $\pi_n^{-1}(C) = \bigcup_{x_i \in I_i}\{x_i\} \times h_{x_i}^{-1}(C_{x_i})$. If m_{n-1} and m_{n-2} denote the $(n-1)$-dimensional and $(n-2)$-dimensional Lebesgue measures respectively, then:

$$\begin{aligned}\mathrm{Vol}_{n-1}(C) &= \int_C dm_{n-1} = \sqrt{n}\int_{\pi_n^{-1}(C)} dm_{n-1} = \sqrt{n}\int_{a_i}^{b_i}\Big(\int_{h_{x_i}^{-1}(C_{x_i})} dm_{n-2}\Big)dx_i \\ \mathrm{Vol}_{n-2}(C_{x_i}) &= \int_{C_{x_i}} dm_{n-2} = \sqrt{n-1}\int_{h_{x_i}^{-1}(C_{x_i})} dm_{n-2}.\end{aligned}$$

Combining these two expressions we obtain that $\mathrm{Vol}_{n-1}(C) = \frac{\sqrt{n}}{\sqrt{n-1}}\int_{a_i}^{b_i}\mathrm{Vol}_{n-2}(C_{x_i})dx_i$, or, equivalently, $\int_{a_i}^{b_i} g_i(E,E-x_i)dx_i = 1$. Now, if $j \neq i$, then $\int_C x_j dm_{n-1} = \frac{\sqrt{n}}{\sqrt{n-1}}\int_{a_i}^{b_i}\Big(\int_{C_{x_i}} x_j dm_{n-2}\Big)dx_i$. Therefore,

$$\begin{aligned}\mu_j(C) &= \frac{1}{\mathrm{Vol}_{n-1}(C)}\int_C x_j dm_{n-1} = \frac{1}{\mathrm{Vol}_{n-1}(C)}\frac{\sqrt{n}}{\sqrt{n-1}}\int_{a_i}^{b_i}\Big(\int_{C_{x_i}} x_j dm_{n-2}\Big)dx_i \\ &= \frac{1}{\mathrm{Vol}_{n-1}(C)}\frac{\sqrt{n}}{\sqrt{n-1}}\int_{a_i}^{b_i}\mu_j(C_{x_i})\,\mathrm{Vol}_{n-2}(C_{x_i})dx_i = \int_{a_i}^{b_i}\mu_j(C_{x_i})g_i(E,E-x_i)dx_i,\end{aligned}$$

because, by definition, $\mu_j(C_{x_i}) = \frac{1}{\mathrm{Vol}_{n-2}(C_{x_i})}\int_{C_{x_i}} x_j dm_{n-2}$. □

Now, we can prove that the core-center rule is a differentiable function of the endowment. For each $j \in N$ denote $\chi_j(E,d) = 0$ if $E < d_j$ and $\chi_j(E,d) = 1$ otherwise.

Theorem D.2. *Let* $d = (d_1,\ldots,d_n) \in \mathbb{R}^N$ *be a vector of claims such that* $0 < d_1 \leq \cdots \leq d_n$. *If* $|N| \geq 3$ *then* $\mu(\cdot,d)$ *is a continuously differentiable function on* $[0,d(N)]$. *Moreover,*

1. *if* $E \in [0,d_1]$ *then* $\frac{\partial \mu_j}{\partial E}(E,d) = \frac{1}{n}$ *for all* $j \in N$.
2. *if* $E \in [d(N\backslash\{n\}),d_n]$ *then* $\frac{\partial \mu_j}{\partial E}(E,d) = 0$ *for all* $j \in N\backslash\{n\}$ *and* $\frac{\partial \mu_n}{\partial E}(E,d) = 1$.

[1] If $i = n$ take $h_{x_n}(x_2,\ldots,x_n) = (E-x_2-\cdots-x_n,x_2,\ldots,x_n)$.

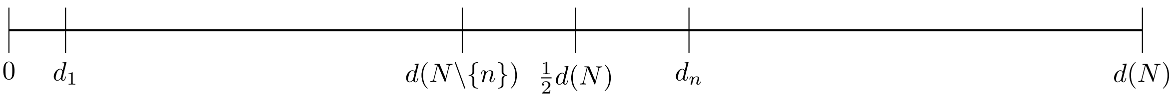
3. if $E \in \big[d_1, \min\{\frac{1}{2}d(N), d(N\backslash\{n\})\}\big]$ *then*

$$\frac{\partial \mu_j}{\partial E}(E,d) = g_n(E,E)\big(\mu_j(E,d_{-n}) - \mu_j(E,d)\big) + \chi_n(E,d) g_n(E,E-d_n)\big(\mu_j(E,d) - \mu_j(E-d_n,d_{-n})\big), \text{ for all } j \in N\backslash\{n\}$$

$$\frac{\partial \mu_n}{\partial E}(E,d) = g_1(E,E)\big(\mu_n(E,d_{-1}) - \mu_n(E,d)\big) + g_1(E,E-d_1)\big(\mu_n(E,d) - \mu_n(E-d_1,d_{-1})\big)$$

4. if $E \in [\frac{1}{2}d(N), d(N)]$ *then* $d(N) - E \in [0, \frac{1}{2}d(N)]$ *and* $\frac{\partial \mu_j}{\partial E}(E,d) = \frac{\partial \mu_j}{\partial E}(d(N)-E,d)$ *for all* $j \in N$.

Proof. If $E \in [0, d_1]$ we know from Proposition B.1 that $\frac{\partial \mu_j}{\partial E}(E,d) = \frac{1}{n}$ for all $j \in N$. If $E \in [d(N\backslash\{n\}), d_n]$, again from Proposition B.1, we conclude that $\frac{\partial \mu_j}{\partial E}(E,d) = 0$ for all $j \in N\backslash\{n\}$ and $\frac{\partial \mu_n}{\partial E}(E,d) = 1$. Next, let us prove that $\mu(\cdot,d)$ is differentiable on the interval $\big[d_1, \min\{\frac{1}{2}d(N), d(N\backslash\{n\})\}\big]$. We distinguish two cases. Case 1: If $d(N\backslash\{n\}) < \frac{1}{2}d(N)$.



Take $E \in [d_1, d(N\backslash\{n\})]$. Then $r_n(E,d) = \max\{0, E-d_n\} = 0$ and $R_n(E,d) = \min\{E, d(N\backslash\{n\})\} = E$ so, by Theorem D.1, we have that $\mu_j(E,d) = \int_0^E \mu_j(u,d_{-n}) g_n(E,u)du$ for all $j \in N\backslash\{n\}$. Recall that g_n is a continuous differentiable function, $\frac{\partial g_n}{\partial E}$ is continuous, and $\frac{\partial g_n}{\partial E}(E,u) = -\frac{\sqrt{n}}{\sqrt{n-1}} \frac{V(u,d_{-n})}{(V(E,d))^2} \frac{\partial V}{\partial E}(E,d)$. Now, applying Leibniz's rule for differentiation under the integral sign and using the expression for $\frac{\partial V}{\partial E}(E,d)$ given in Theorem A.4, we obtain that $\mu_j(\cdot,d)$ is differentiable at E for all $j \in N\backslash\{n\}$ and

$$\begin{aligned}
\frac{\partial \mu_j}{\partial E}(E,d) &= \int_0^E \mu_j(u,d_{-n}) \frac{\partial g_n}{\partial E}(E,u)du + \mu_j(E,d_{-n}) g_n(E,E) \\
&= -\frac{n}{n-1} \int_0^E \mu_j(u,d_{-n}) \frac{V(u,d_{-n})V(E,d_{-n})}{\big(V(E,d)\big)^2} du + \mu_j(E,d_{-n}) \frac{V(E,d_{-n})}{V(E,d)} \frac{\sqrt{n}}{\sqrt{n-1}} \\
&= \frac{\sqrt{n}}{\sqrt{n-1}} \frac{V(E,d_{-n})}{V(E,d)} \Big(-\int_0^E \mu_j(u,d_{-n}) \frac{V(u,d_{-n})}{V(E,d)} \frac{\sqrt{n}}{\sqrt{n-1}} du + \mu_j(E,d_{-n} \Big) \\
&= g_n(E,E)\big(\mu_j(E,d_{-n}) - \mu_j(E,d)\big).
\end{aligned}$$

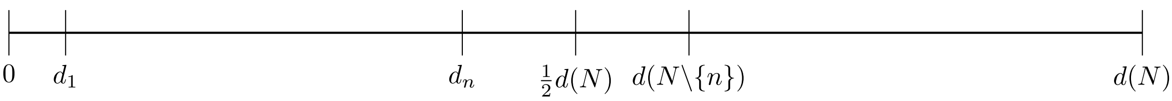
Since $r_1(E,d) = E - d_1$ and $R_1(E,d) = E$, from Theorem D.1, $\mu_n(E,d) = \int_{E-d_1}^E \mu_n(u,d_{-1}) g_1(E,u)du$. Applying the Leibniz's rule and the chain rule we have that $\mu_n(\cdot,d)$ is differentiable at E and

$$\frac{\partial \mu_n}{\partial E}(E,d) = \int_{E-d_1}^E \mu_n(u,d_{-1}) \frac{\partial g_1}{\partial E}(E,u)du + \mu_n(E,d_{-1}) g_1(E,E) - \mu_n(E-d_1,d_{-1}) g_1(E,E-d_1).$$

From Theorem A.4 we know that $\frac{\partial g_1}{\partial E}(E,u) = -\frac{n}{n-1} \frac{V(u,d_{-1})\big(V(E,d_{-1}) - V(E-d_1,d_{-1})\big)}{\big(V(E,d)\big)^2}$. Then,

$$\begin{aligned}
\frac{\partial \mu_n}{\partial E}(E,d) &= -\mu_n(E,d)(g_1(E,E) - g_1(E,E-d_1)) + \mu_n(E,d_{-1}) g_1(E,E) - \mu_n(E-d_1,d_{-1}) g_1(E,E-d_1) \\
&= g_1(E,E)\big(\mu_n(E,d_{-1}) - \mu_n(E,d)\big) + g_1(E,E-d_1)\big(\mu_n(E,d) - \mu_n(E-d_1,d_{-1})\big).
\end{aligned}$$

Case 2: If $d(N\backslash\{n\}) > \frac{1}{2}d(N)$.



If $E \in [d_1, d_n]$ then $r_n(E,d) = 0$, $R_n(E,d) = E$, $r_1(E,d) = E - d_1$ and $R_1(E,d) = E$. By Theorem D.1, $\mu_j(E,d) = \int_0^E \mu_j(u, d_{-n}) g_n(E,u)du$ for all $j \in N\backslash\{n\}$ and $\mu_n(E,d) = \int_{E-d_1}^{E} \mu_n(u, d_{-1}) g_1(E,u)du$. Applying Leibniz's rule as in the previous case we conclude that $\mu(\cdot, d)$ is differentiable at E and we obtain the same expressions for the derivatives $\frac{\partial \mu_j}{\partial E}(E,d)$ for all $j \in N$. Finally, If $E \in [d_n, \frac{1}{2}d(N)]$ then $r_n(E,d) = E - d_n$, $R_n(E,d) = E$, $r_1(E,d) = E - d_1$ and $R_1(E,d) = E$. Thus, $\mu_j(E,d) = \int_{E-d_n}^{E} \mu_j(u, d_{-n}) g_n(E,u)du$ for all $j \in N\backslash\{n\}$ and $\mu_n(E,d) = \int_{E-d_1}^{E} \mu_n(u, d_{-1}) g_1(E,u)du$. As above, the Leibniz's rule allows us to assert that $\mu(\cdot, d)$ is differentiable at $[d_n, \frac{1}{2}d(N)]$ and to compute its derivatives as above.

Finally, we know that $\mu(\cdot, d)$ is a continuous function on $[0, d(N)]$. We have seen that $\mu(\cdot, d)$ is also a differentiable function on $[0, \frac{1}{2}d(N)]$ except perhaps at the points d_1, d_n and $d(N\backslash\{n\})$. It is easy to check that, in fact, $\mu(\cdot, d)$ is also differentiable at those points. Therefore $\mu(\cdot, d)$ is differentiable on $[0, \frac{1}{2}d(N)]$. But, since the core-center rule satisfies self-duality, if $E \in [\frac{1}{2}d(N), d(N)]$ then $d(N) - E \in [0, \frac{1}{2}d(N)]$ and $\mu(E,d) = d - \mu(d(N) - E, d)$, so $\mu(\cdot, d)$ is differentiable at E and $\frac{\partial \mu_j}{\partial E}(E,d) = \frac{\partial \mu_j}{\partial E}(d(N) - E, d)$ for all $j \in N$. □

E Core-center rule bounds

Certainly, the core-center rule satisfies $\frac{1}{|N|}$-truncated-claims lower bounds on awards when $E \in [0, d_1]$. Now, we show that this is also the case if $E \in [d_1, d_2]$.

Lemma E.1. *Let $(E,d) \in B^N$ be a bankruptcy problem such that $0 \le d_1 \le \cdots \le d_n$. If $d_1 < E \le d_2$ then $\mu_1(E,d) \ge \frac{d_1}{n}$ and $\mu_j(E,d) \ge \frac{E}{n}$ for all $j \in N\backslash\{1\}$.*

Proof. It is straightforward from equation (1) that the result holds when $|N| = 2$. By Theorem D.1 we have that

$$\mu_1(E,d) = \int_0^E \mu_1(u, d_{-n}) g_n(E,u)du = \int_0^{d_1} \mu_1(u, d_{-n}) g_n(E,u)du + \int_{d_1}^{E} \mu_1(u, d_{-n}) g_n(E,u)du. \tag{2}$$

But

$$\begin{aligned}\int_0^{d_1} \mu_1(u, d_{-n}) g_n(E,u)du &= \int_0^{d_1} \mu_1(u, d_{-n}) \frac{V(u, d_{-n})}{V(E,d)} \frac{V(d_1, d)}{V(d_1, d)} \frac{\sqrt{n}}{\sqrt{n-1}} du \\ &= \frac{V(d_1,d)}{V(E,d)} \int_0^{d_1} \mu_1(u, d_{-n}) g_n(d_1, u)du = \frac{V(d_1,d)}{V(E,d)} \mu_1(d_1, d) = \frac{V(d_1,d)}{V(E,d)} \frac{d_1}{n}. \end{aligned} \tag{3}$$

Let $u \in (d_1, E]$. Since the core-center rule satisfies endowment monotonicity we know that $\mu_1(u, d_{-n}) \ge \mu_1(d_1, d_{-n}) = \frac{d_1}{n-1}$ and, from Theorem A.4, $\frac{\partial V}{\partial t}(u,d) = \frac{\sqrt{n}}{\sqrt{n-1}} V(u, d_{-n})$. Therefore,

$$\begin{aligned}\int_{d_1}^{E} \mu_1(u, d_{-n}) g_n(E,u)du &\ge \frac{d_1}{n-1} \int_{d_1}^{E} g_n(E,u)du = \frac{d_1}{n-1} \frac{1}{V(E,d)} \int_{d_1}^{E} \frac{\sqrt{n}}{\sqrt{n-1}} V(u, d_{-n})du \\ &= \frac{d_1}{n-1} \frac{1}{V(E,d)} \Big(V(E,d) - V(d_1, d) \Big) = \frac{d_1}{n-1} \Big(1 - \frac{V(d_1,d)}{V(E,d)} \Big). \end{aligned} \tag{4}$$

Combining (2), (3), and (4), and since $V(d_1, d) \le V(E,d)$, we have that

$$\begin{aligned}\mu_1(E,d) &\ge \frac{d_1}{n} \frac{V(d_1,d)}{V(E,d)} + \frac{d_1}{n-1} \Big(1 - \frac{V(d_1,d)}{V(E,d)} \Big) = \frac{d_1}{n-1} - \frac{d_1}{n(n-1)} \frac{V(d_1,d)}{V(E,d)} \\ &\ge \frac{d_1}{n-1} - \frac{d_1}{n(n-1)} = \frac{d_1}{n}. \end{aligned}$$

Finally, since the core-center rule satisfies order preservation of awards we have that $\mu_n(E,d) \ge \frac{E}{n}$. But then, by Proposition B.1, $\mu_j(E,d) = \mu_n(E,d) \ge \frac{E}{n}$ for all $j \in N\backslash\{1\}$. □

Let us see that if the core-center rule satisfies $\frac{1}{|N|}$-truncated-claims lower bounds on awards for any bankruptcy problem for which the initial endowment lies between the claims $i-1$ and i then it satisfies $\frac{1}{|N|}$-truncated-claims lower bounds on awards for any bankruptcy problem for which the initial endowment lies between the claims i and $i+1$.

Lemma E.2. *Let $i \in N\backslash\{1\}$ and let $d, c \in \mathbb{R}^N$ such that $0 \leq d_1 \leq \cdots \leq d_n$ and $0 \leq c_1 \leq \cdots \leq c_n$. If $\mu_j(E', c) \geq \frac{1}{n}\min\{E', c_j\}$ for all $j \in N$ and $E' \in [c_{i-1}, c_i]$ then $\mu_j(E, d) \geq \frac{1}{n}\min\{E, d_j\}$ for all $j \in N$ and $E \in [d_i, d_{i+1}]$.*

Proof. Let $j \in N\backslash\{i\}$ and $E \in [d_i, d_{i+1}]$. Then, since the core-center rule satisfies other-regarding claim monotonicity $\mu_j(E, d) = \mu_j\big(E, (d_1, \ldots, d_i, E, \ldots, E)\big) \geq \mu_j\big(E, (d_1, \ldots, d_{i-1}, E, \ldots, E)\big)$. But, from the assumption, we have that $\mu_j\big(E, (d_1, \ldots, d_{i-1}, E, \ldots, E)\big) \geq \frac{1}{n}\min\{E, d_j\}$ so $\mu_j(E, d) \geq \frac{1}{n}\min\{E, d_j\}$. On the other hand, applying other-regarding claim monotonicity and anonymity,

$$\begin{aligned}\mu_i(E, d) &= \mu_i\big(E, (d_1, \ldots, d_i, E, \ldots, E)\big) \geq \mu_i\big(E, (d_1, \ldots, d_{i-2}, E, d_i, E, \ldots, E)\big)\\ &= \mu_{i-1}\big(E, (d_1, \ldots, d_{i-2}, d_i, E, \ldots, E)\big).\end{aligned}$$

By the hypothesis, $\mu_{i-1}\big(E, (d_1, \ldots, d_{i-2}, d_i, E, \ldots, E)\big) \geq \frac{1}{n}\min\{E, d_i\}$ so $\mu_i(E, d) \geq \frac{1}{n}\min\{E, d_i\}$. $\square$