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Market games and Walrasian equilibria

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Abstract. In this work, we recapitulate and compare the market game approaches provided by Shapley and Shubik (1977) and Schmeidler (1980). We provide some extensions to economies with infinitely many commodities and point out some applications and lines for future research.

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1 Introduction

The seminal work of Nash (1950) founded the genesis for a rapidly growing series of papers on strategic approaches to economic equilibrium. Debreu (1952) and Arrow and Debreu (1954) obtained existence of Walrasian equilibrium of an economy as Nash equilibrium of an associated generalized game, where the players are the consumers of the original economy in addition to a fictitious player or auctioneer selecting prices and whose payoff is given by the value of the aggregate excess demand. This equilibrium existence proof relies on Kakutani's fixed point theorem and does not provide any insight on how Walrasian allocations and prices are formed.

Walrasian equilibria are related to cooperative solutions as well. In fact, Debreu and Scarf (1963) formalized Edgeworth's conjecture by characterizing the set of Walrasian allocations as the intersection of the cores of a sequence of replicated economies. This core convergence result was strengthened by Auman (1964) through the core-Walras equivalence showing that the core coincides with the set competitive allocations for atomless economies. In both characterizations, equilibrium prices are obtained by applying the separation theorem of convex sets. Cooperative approaches to the notion of perfect competition have been a major focus of research in mathematical economics since the 1970's to our days providing deep theoretical foundations to the Walrasian paradigm. Notable contributions in this direction include Arrow and Hahn (1971), Bewley (1973), Hildenbrand (1974), Dierker (1975), Khan (1976), Anderson (1978, 1981, 1985), Ostroy and Zame (1994), Wooders (1994), Tourky and Yannelis (2001), Podczeck (2005) and Greinecker and Podczeck (2017). Following these cooperative descriptions of perfect competition, equilibrium existence is established through a coalition formation mechanism and, as before, does not elucidate the configuration of the pricing system.

Within a general equilibrium setting, we also find a variety of games that characterize the equilibrium of the economy as a non-cooperative solution of a strategic market game or a generalized game. Although most of them consider consumers, and/or firms, as players, Hervés-Beloso and Moreno-García (2009) define an associated game with only two players, regardless of the number of consumers, where the whole society representing all the agents in the economy plays two different roles: as player 1, it is a Paretian player that aims efficiency, and as player 2, it aims for a fair behavior against the Paretian player. Under the stated assumptions one has existence of Walrasian equilibrium and it is shown that the set of Walrasian allocations coincides with the strong Nash equilibria of the game.

Further game theoretical analysis for the consumers behavior in the markets, that aims to explain both exchange and price-setting processes in addition to the consumer behavior in the market constitute well known alternatives to the Walrasian model. The wide literature on market games uses the principles of game theory to motivate or justify the description of markets in which certain behavioral characteristics, such as price-taking behavior, are assumed. Most of these works show how strategic interactions by rational agents lead to a competitive equilibrium situation. Game theoretic approaches to market solutions (in particular, to Walrasian or competitive equilibrium) provide insights into the market mechanism through which trade is conducted. One of the advantages of building strategic foundations for perfect competition is that a complete description of the process how the equilibrium allocations and prices are reached becomes necessary.

Most of the research on market games includes the following three steps; firstly, describe the market or the whole economy; secondly, define an extensive-form game describing the behavior of the agents in the market or in the economy; and thirdly, analyze the market game to show the relation of the solutions of the game with the equilibrium of the original economy. As it is not surprising, there are many ways in which this program can be carried out. Actually, strategic market games may be classified into different categories depending basically on the underlying strategy sets for players and on the way in which every agent's signal is used to determine market prices.

Many market games can be viewed as extensions of the single market analysis of Cournot (1838) and Bertrand (1883) to multiple markets within a general equilibrium framework. The extension of the Cournot tradition to general equilibrium was pioneered by the works by Shubik (1973), Shapley (1976), Shapley and Shubik (1977), and Dubey and Geanokoplos (2003). The papers by Hurwicz (1979), Schmeidler (1980) and Dubey (1982) followed the Bertrand tradition. See Giraud (2003) for a complete survey on market games.

In Sections 2 and 3 we focus, respectively, on the work by Shapley and Shubik (1977) followed by Dubey and Geanakoplos (2003) to provide a more direct route from Nash to Walras, and the work by Schmeidler (1980). Our aim is to analyze the two approaches and the corresponding main results that show how the Walrasian equilibrium may be regarded either as the limit solution or outcome for non-cooperative notions of equilibrium. In Section 4, we compare the differences regarding the formulation of the games and their implications. Following Bewley (1972) and Araujo (1985), in Section 5, we consider an economy with infinitely many commodities to show that any Walrasian equilibrium of the economy can be attained as a Nash equilibrium of the associated market game. Finally, we summarize some applications that have been analyzed addressing a variety of settings and we point out some lines of future research.

2 Shapley-Shubik's market game

Shapley and Shubik (1977) provide a market game describing a general model of noncooperative trading equilibrium that avoids the assumption that individuals must regard prices as fixed. Actually, in a natural way prices depend on the buying and selling decisions of the traders and the key to the approach is the use of a single, specified commodity as "cash," which may or may not have intrinsic value. The rules of the game that include the price-forming mechanism, are independent of behavioral or equilibrium assumptions, which enter, instead, through the solutions of the game. The model, in several variants, is a noncooperative game, in the spirit of Nash and Cournot.

In the basic model there are n traders trading in $m+1$ goods, where the $(m+1)th$ good has a special operational role in addition to its possible utility in consumption.

Each trader $i \in N = \{1, \ldots, n\}$ is characterized by an initial bundle of goods, $a^i = (a^i_1, a^i_2, \ldots, a^i_m, a^i_{m+1})$, and a concave utility function, $U_i : \mathbb{R}^{m+1}_+ \to \mathbb{R}$. Although it is considered that U_i depends on $(x^i_1, x^i_2, \ldots, x^i_m, x^i_{m+1})$, we emphasize that U_i need not actually depend on x^i_{m+1}; the possibility of a fiat money is not excluded.

Let us imagine m separate trading posts, one for each of the first m commodities, where the total supplies $(\bar{a}_1, \ldots, \bar{a}_m)$, with $\bar{a}_j = \sum_{i=1}^n a^i_j, j = 1, \ldots, m$, assumed to be positive, have been deposited for sale "on consignment."

Each trader makes bids by allocating amounts of his $(m+1)th$ commodity among the m trading posts. Thus, the strategy set for trader (or player) i is:

$$S_i = \left\{ b^i = (b^i_1, \ldots, b^i_m), \text{ such that } b^i_j \geq 0, \; j = 1, \ldots, m \text{ and } \sum_{j=1}^{m} b^i_j \leq a^i_{m+1} \right\}$$

The price formation rule and the allocation mechanism are as follows. For each strategy profile $b = (b^1, \ldots, b^n)$ and each commodity $j = 1, \ldots, m$, let $p_j(b)$ be defined by

$$p_j(b) = \frac{\bar{b}_j}{\bar{a}_j}, \text{ where } \bar{b}_j = \sum_{i=1}^{n} b^i_j.$$

Now for every trader i consider the bundle $x^i(b)$ given by

$$x^i_j(b) = \begin{cases} b^i_j / p_j(b) & \text{if } p_j(b) > 0 \\ 0 & \text{if } p_j(b) = 0 \end{cases} \qquad \text{for each } j = 1, \ldots, m; \text{ and}$$

$$x^i_{m+1}(b) = a^i_{m+1} - \textstyle\sum_{j=1}^{m} b^i_j + \sum_{j=1}^{m} p_j a^i_j.$$

The payoff function for player i is $\Pi_i(b) = U_i(x^i(b))$, that is

$$\Pi_i(b^1, \ldots, b^n) = U_i(x^i_1(b), \ldots, x^i_m(b), x^i_{m+1}(b))$$

Given a strategy profile b, let b_{-i} denote the strategies of all players except i. A Nash equilibrium is a profile b^* such that $\Pi_i(b^*) \geq \Pi_i(b^*_{-i}, s)$ for every $s \in S_i$ and every player i.

Shapley and Shubik (1977) obtain the following results. The first theorem states existence of equilibrium for the market game and the second one is a convergence theorem that relates the equilibria of the game to the competitive equilibria.

Theorem 1. *For each trader $i = 1, \ldots, n$, let Ui be continuous, concave, and nondecreasing. For each good $j = 1, ..., m$, let there be at least two traders with positive initial endowments of good $m + 1$ whose utility for good j is strictly increasing. Then a Nash equilibrium exists.*

Note that there is no assumption that good $m+1$ has intrinsic value to anyone. It must merely be available to large enough number of agents so that nontrivial markets for the other goods can be formed.

Let $(r\mathcal{E}, r \in \mathbb{N})$ be the sequence of replicated economies, being $r\mathcal{E}$ the economy with r agents of each type $i = 1, \ldots, n$. A trader of type i is characterized by endowments a^i and the utility function U_i.

Theorem 2. *Assume that for infinitely many values of r the market has a symmetric, interior Nash equilibrium and let p^r be the corresponding m-dimensional vector of prices. Let p be any limit point of the sequence p^r and define $p_{m+1} = 1$. Then the $m+1$ prices $(p_1, \ldots, p_m, p_{m+1})$ will be competitive for the market (for any value of r).*

It should be noted that the Nash equilibrium approaches the competitive equilibrium "from below," that is, through outcomes that are not in general Pareto optimal. This contrasts with the convergence of cooperative solutions like the core and the value, which are, by definition, Pareto optimal all the way. Another drawback of this approach is that, in general, there may be not enough of the *numéraire* commodity or of money to sustain all the possible competitive trades.

To propose a more direct route from Nash to Walras, Dubey and Geanakoplos (2003) consider a variant of the Shapley-Shubik trading-post game. They start form a pure exchange economy where agents, initially have no money ($a_{m+1} = 0$), but can borrow up to certain units at zero interest from a bank and choose how much to bid at each trading-post for purchases. This inside fiat money is the sole medium of exchange and it must be repaid to the bank after trade. To trigger the trade, an external agent also bids one dollar at each trading post. The bank, the external agent and the trading-posts are all assumed to be strategic dummies.

To simplify the reasoning, they consider a continuum of players with a finite number n of different types. However, their argument works as well when the finite number of agents of each type increases. As in Shapley-Shubik's game, for every commodity $j = 1, \ldots, m$, there is a "trading-post" and agents put up their entire endowment of that commodity for sale and (fiat) money for purchase.

For each fixed amount of money M, a game $\Gamma(M)$ in normal form is defined. In this game, the strategy set for each player is

$$\Delta_M = \left\{ b = (b_1, \ldots, b_m), \text{ such that } b_j \geq 0,\ j = 1, \ldots, m \text{ and } \textstyle\sum_{j=1}^m b_j \leq M \right\}$$

Assuming that agents are representatives of their type, that is, all agents of type i choose the same strategy $b^i \in \Delta_M$, the (type-symmetric) strategy profile

$b = (b^i)_{i\in N}$ defines a price system blue $p(b)$, where the price of commodity j is given by

$$p_j(b) = \frac{\bar{b}_j + 1}{\bar{a}_j}, \text{ where } \bar{b}_j = \sum_{i=1}^{n} b^i_j.$$

Each player of type i obtains the consumption bundle $x^i = (x^i_j(b), j = 1, \ldots, m)$,where $x^i_j(b) = b^i_j/p_j(b)$, and also obtains $p(b)a^i$ units of money as revenue from the sale at prices $p(b)$ of their endowments, leaving them with the surplus or net deficit $d^i(b) = \sum_{j=1}^{m} b^i_j - \sum_{j=1}^{m} p_j(b)a^i_j$.

The payoff of agents of type i, for symmetric profiles, is given by $\Pi_i(b) = u_i(x^i(b)) - \max\{0, d^i(b)\}$.

The max term reflects the fact the agents gain no utility from fiat money, but are penalized from defaulting on their loans.

In the game $\Gamma(M)$, prices mediate trade, trading-posts clear and generate a feasible reallocation of the endowments, independently of what the agents bid. Moreover, if each agent optimizes, given the strategies of the others, a Nash equilibrium is obtained.

In fact, the first result in Dubey-Geanakoplos (2003) shows the existence of a type-symmetric Nash equilibria (TSNE) of $\Gamma(M)$, assuming that each agent has strictly positive endowments and their utility function is weakly monotone, continuous and concave.

The set of Nash equilibria of $\Gamma(M)$ may depend on the bound M of fiat money that agents can borrow from the bank. To allow all the competitive tradings, for each natural number M, let b_M be a TSNE of the game $\Gamma(M)$. Dubey-Geanakoplos (2003) showed that the sequence of allocations $x_M = (x^i(b_M))_{i\in N}$ and prices $p_M = \frac{p(b_M)}{\|p(b_M)\|}$ is uniformly bounded and then, there is a subsequence converging to (x, p). Moreover, their main result states that the limit point (x, p) is a Walrasian equilibrium of the exchange economy. Thus, for M big enough, Nash equilibria of $\Gamma(M)$ approximate Walrasian equilibrium of the initial exchange economy.

3 Schmeidler's market game

Schmeidler (1980) provides a rigorous description of a game in a strategic form whose Nash equilibria are all strong equilibria coinciding with the Walras equilibria of the underlying Arrow-Debreu pure exchange economy.

Consider the economy $\mathcal{E}$ with n agents and ℓ commodities. Each agent i is endowed with the bundle $\omega_i \in \mathbb{R}^\ell_+$ and has a preference relation $\succeq_i$ represented by a strictly quasi-concave increasing utility function $U_i : \mathbb{R}^\ell_+ \to \mathbb{R}$.

Given the economy $\mathcal{E}$, Schmeidler (1980) considers an associated game $\mathcal{G}$ where each of the n consumers is represented by a player.[1] A strategy for a player is a consumption bundle and a price vector[2] such that the bundle she chooses belongs to her budget set at the announced prices. That is, the strategy set for player i is:

$$S_i = \{(x, p) |\ p \cdot x \leq p \cdot \omega_i\}.$$

A strategy profile s is given by a strategy s_i for every player $i \in N = \{1, \ldots, n\}$. We denote $s = (s_1, s_2, \ldots, s_n) = ((x_1, p_1), (x_2, p_2), \ldots, (x_n, p_n))$.

Schmeidler's (1980) proposal is crystal: agents trade only if they agree on the prices. Following this idea, given a strategy profile, each player trades only with those individuals that select the same price. Thus, for each profile $s = ((x_1, p_1), (x_2, p_2), \ldots, (x_n, p_n))$, let $A_i(s) = \{j \in I = \{1, \cdots, n\}\ \ p_j = p_i\}$ and let $\# A_i(s)$ denote the cardinality of the set $A_i(s)$, i.e, the number of players that choose the price selected by the ith one in the profile s. Then, the average excess of demand of players in $A_i(s)$ is

$$\gamma_i(s) = \frac{1}{\# A_i(s)} \sum_{j \in A_i(s)} (x_j - \omega_j),$$

Each player receives the bundle they choose adjusted by the average excess of demand of the players that proposes the same price. That is, given s, the player

[1] To describe the game and the main result we do not use the same notation that appears in Schmeidler (1980) since we state the game in terms of trades whereas in Schmeidler's paper it is written in terms of net trade instead.

[2] The set of prices is the the simplex $\{p \in \mathbb{R}^\ell_+ |\ \sum_{h=1}^{\ell} p_h = 1\}$.

i gets the bundle

$$f_i(s) = x_i - \gamma_i(s) = x_i - \frac{\sum_{j \in A_i(s)} (x_j - \omega_j)}{\# A_i(s)}.$$

Finally, the payoff function for player i is $\Pi_i(s) = U_i(f_i(s))$.

The strategy profile s^* is a Nash equilibrium if no player has incentives to deviate individually, i.e., $\Pi_i(s^*) \geq \Pi_i(s^*_{-i}, s_i)$, for every $s_i \in S_i$ and every i.

The profile s^* is a strong Nash equilibrium if no coalition of players has incentives to modify their strategies as a group.

The main result proved by Schmeidler (1980) is the next theorem.

Theorem. *Let $\mathcal{E}$ be an economy with $n \geq 3$. Let $\mathcal{G}$ be the associated game. The following statements hold:*

(i) If (x^*, p^*) is Walrasian equilibrium of the economy $\mathcal{E}$, then $s^* = ((x^*_i, p^*)_{i \in N}$ is a strong Nash equilibrium of $\mathcal{G}$.

(ii) If s^* is a Nash equilibrium of the game $\mathcal{G}$, all the players choose the same price p^* and $((f_i(s^*)_{i \in N}, p^*)$ is an equilibrium of the economy $\mathcal{E}$.

The proof of the above result follows several steps showing that if s^* is a Nash equilibrium of the game $\mathcal{G}$ then

- Given two different players i, j, we have $f_i(s^*) \succsim_i d_i(p_j)$, where d_i denotes the demand function of agent i.
- If at least one more trader selects the same price p as player i, then $f_i(s^*) = d_i(p)$.
- If $\#A_i(s^*) \geq 2$ for some i, then $A_i(s^*) = \{1, \ldots, n\}$ and all of them get the demand at the chosen price.
- $\#A_i(s^*) > 1$ for some i.

The most significant drawback of Schmeidler's (1980) approach is the non-feasibility of the individual allocations for some strategy profiles. In fact, it could happen that, for a profile s, the commodity bundle $f_i(s)$ does not belong to $\mathbb{R}^\ell_+$.

Schmeidler argues that the possibility of individual nonfeasibility is attributed to the total informational decentralization of the model and, in addition, a strategy profile that induces a nonfeasible allocation occurs only out of equilibrium.

4 Shapley-Shubik vs. Schmeidler's market game

The market game approach that Shapley and Shubik (1977) proposed differs from the one that Schmeidler (1980) stated regarding not only the own definition of the game but also the main results that relate the equilibria of the game with the equilibria of the underlying economy.

Shapley and Shubik's trading-post game presents the following characteristics:

(a) It is an extension of the Cournot tradition to general equilibrium where money is explicitly introduced as the stipulated medium of exchange. Although the treatment of money in a strategic market game has been a subject of intense debate, it was described by Shapley (1976) in these terms:

The decisive step was to meet the problem of money head on – to accept the proposition that, in the world of buying and selling, money is "real." Granting this, the rest falls into place with remarkably few other generality-restricting assumptions.

(b) The strategies of each player are "bids" and neither prices nor commodity bundles appears as elements of the strategy sets.

(c) A map assigning prices and feasible reallocations (outcomes) to the agents' strategies (bids) is defined. That is, the game provide a price formation mechanism and an assignment process. In other words, agents are assumed to send quantity-setting strategies to trading posts, where prices form so as to equalize supply and demand on each market. Moreover, no matter what strategies agents choose, a feasible outcome is always engendered

(d) The first main result is existence of Nash equilibrium for the market game. Moreover, it is shown that if the market has symmetric and interior Nash equilibria, the sequence of Nash equilibria associated with replicated economies converge to the Walrasian equilibria, whenever there is a limit point of the corresponding sequence of prices. That is, noncooperative equilibrium exists, and as the number of agents increases, under the previous assumptions, price-taking behavior is induced and Walrasian equilibrium is achieved in the limit.

In contrast, the market game that Schmeidler (1980) introduced presents the following features:

(a) It extends the single market analysis of Bertrand to multiple markets within a general equilibrium framework.

(b) The strategy set of every player is a pair formed by a price and a consumption bundle that is in their budget set when the selected prices prevail.

(c) The exchange mechanism that characterizes the economic institutions of trade is given by strategic outcome functions, with players proposing consumption bundles and prices. Thus, the outcome function maps players' simultaneous selections of strategies into allocations. In this way, it is explained the price formation mechanism but there is no explicit price formation rule as in Shaplye-Shubik's game.

(d) The main result shows that Nash equilibria of this market game are strong and coincide with the Walrasian equilibria of the underlying Arrow-Debreu pure exchange economy. In this case, the existence of Nash equilibrium relies on the existence of Walrasian equilibrium, rather than the other way around.

We remark that in both aforementioned market game no price player is involved, nor are generalized games. Each of the two approaches gives rise to a different non-cooperative game in strategic form and focuses on features of strategic (Nash) equilibria and their relation to competitive (Walras) equilibria.

5 An extension to infinitely many commodities

In this section, following Bewley (1972), and more closely Araujo (1985), we consider the economy $\mathcal{E} = (\ell_\infty^+, \succsim_i, \omega_i)_{i=1,\ldots,n}$ where the commodity space ℓ_∞ is the Banach space of bounded sequences of real numbers, representing a model where a consumption plan is a sequence of points in $\mathbb{R}^\ell_+$ for each time period

$t = 1, 2, \ldots$. Each consumer i is characterized by a preference relation $\succsim_i$ defined on the consumption set ℓ_∞^+ and by endowments $\omega_i \in \ell_\infty^+$.

A price system is an element of the dual space of ℓ_∞, denoted by ℓ'_∞. A Walrasian equilibrium is a pair $((x_1, \ldots x_n), p) \in (\ell_\infty^+)^n \times \ell'_\infty$, with $p \neq 0$, such that $\sum_{i=1}^n x_i = \sum_{i=1}^n \omega_i$ and, for every i, the consumption plan x_i maximizes $\succsim_i$ on $\{z \in \ell_\infty^+ |\ p \cdot z \leq p \cdot \omega_i\}$.

Under the assumptions of interiority of endowments (i.e., there exists $\varepsilon > 0$ such that $\omega_{i,k} > \varepsilon$ for every natural number k and every $i = 1, \ldots, n$) and convexity, monotonicity and Mackey continuity of preferences, Araujo (1985) showed existence of Walrasian equilibrium prices in $\ell_1 \subset \ell'_\infty$.[3]

Next, let us consider the associated continuum economy with n-types of agents, $\mathcal{E}_c = (I = \bigcup_{i=1}^n I_i, \ell_\infty^+, \succsim_t, \omega_t)_{t \in I}$, where the real interval $I = (0, n]$, with the Lebesgue measure μ, represents the set of consumers. Each $t \in I_i = (i-1, i]$ is characterized by the preference relation $\succsim_t = \succsim_i$ and by endowments $\omega_t = \omega_i$. Under the assumptions on endowments and preferences previously stablished, the economy $\mathcal{E}$ has an equilibrium (x, p), with $p \in \ell_1$. It is easy to see that (x, p) defines an equilibrium (x^*, p^*) for the associated n-types continuum economy $\mathcal{E}_c$, where $p^* = p$ and x^* is the step function defined by $x^*(t) = x_i$ if $t \in I_i$.[4]

Let us consider the competitive equilibrium (x^*, p^*) and define $M_i = p^* \cdot \omega_i = \sum_{k=1}^\infty p_k^*\ \omega_{i,k}$, and $M = \sum_{i=1}^n M_i$. Consider a variant of the game proposed by Dubey and Geanakoplos (2003) with no external agent and where the strategy sets for each consumer of type i is defined by the amount of money M_i. That is, the strategy set for a consumer of type i is $S_i = \{b \in \ell_1^+ |\ \sum_{k=1}^\infty b_k \leq M_i\}$.[5] A strategy profile is given by a selection $b(t) \in S_i$ for every $t \in I_i$ such that $b_k(\cdot)$ is a μ-integrable function for every natural number k. A profile $\beta = (b(t))_{t \in (0,n]}$, leads to a price at the trading post k defined by $p_k(\beta) = \frac{\int_I b_k(t) d\mu(t)}{\sum_{i=1}^n \omega_{i,k}}$, for each $k \in \mathbb{N}$. Since, by assumption, endowments are interior points, there is a positive constant $a > 0$ such that $\sum_{i=1}^n \omega_{i,k} > a$ for all i, k. Then, $\sum_{k=1}^\infty p_k(\beta) = \sum_{k=1}^\infty \frac{\int_I b_k(t) d\mu(t)}{\sum_{i=1}^n \omega_{i,k}} <$

[3] Mackey continuity of preferences implies a myopic behavior of agents regarding future time periods, in the sense that both gains and losses in the distant future are negligible. Araujo (1985) shows that if we consider a larger class of preferences, as those that are continuous with respect to the norm topology, then equilibrium, and even any individually rational Pareto optimal allocation, might fail to exist.

[4] See Garcia-Cutrín and Hervés-Beloso (1993) for further details.

[5] Note that the fact that strategies are elements of ℓ_1^+ is in accordance with the myopic behavior of consumers that comes from the Mackey continuity of preferences.

$\frac{1}{a}\sum_{k=1}^{\infty}\int_I b_k(t)d\mu(t) \leqslant \frac{1}{a}\int_I \sum_{k=1}^{\infty} b_k(t)d\mu(t) \leqslant \frac{M}{a}$. Thus, the price system $p(\beta) \in \ell_1$. Prices $p_k(\beta)$ at each trading post define the allocation that assigns to each consumer $t \in I$ the bundle $x_t(\beta)$ as follows:

$$x_{t,k}(\beta) = \begin{cases} \frac{b_k(t)}{p_k(\beta)} \text{ if } p_k(\beta) > 0 \\ 0 \text{ otherwise} \end{cases}$$

Note that if player t bids $b_k(t) = 0$, the k-th coordinate of the commodity bundle they receive is null and that, if almost every agent t bids $b_k(t) = 0$, all players receive 0. Moreover, it is easy to see that the sequence $(x_{t,k}(\beta))_k$ belongs to ℓ_∞^+ and, in addition, $\int_I x_{t,k}(\beta)d\mu(t) = \frac{\int_I b_k(t)d\mu(t)}{p_k(\beta)} \leqslant \sum_{i=1}^{n}\omega_{i,k} = \int_I \omega_{t,k}d\mu(t)$, for every $k \in \mathbb{N}$. Then, every strategic profile β results in a feasible allocation in the economy $\mathcal{E}_c$.

Apart from the allocation $(x_{t,k}(\beta))_k$, agent t also obtains $p(\beta)\cdot\omega_t$ units of money as sales revenue of their endowments. Thus, after returning the loan, they get $d_t(\beta) = \sum_{k=1}^{\infty} b_k(t) - p(\beta)\cdot\omega_t$, which becomes either a debt or a profit.

We emphasize that ℓ_∞ with the Mackey topology is a separable space and the Mackey continuity of preferences guarantees existence of utility functions $U_i,\ i = 1,\dots,n$ representing each preference relation.[6] The payoff function for each $t \in I_i$ is defined as $\pi_t(\beta) = U_i(x_i(\beta)) - \max\{0, d_t(\beta)\}$.

Let x^* be the equal treatment competitive allocation associated with the equilibrium price p^* previously chosen. Then, one can deduce that the symmetric strategy profile β^*, with $b_k^*(t) = p_k^*\cdot x_{i,k}^*$ for each $k \in \mathbb{N}$ and $t \in I_i$, is a Nash equilibrium of the game defined. Moreover, $p(\beta^*) = p^*$ and $x(\beta^*) = x^*$ That is, the type symmetric Nash equilibrium β^* results in the competitive equilibrium of the economy. To show this, note that if a consumer deviates from β^* then the price is not altered, and the bundle they get belongs to their budget set at such a price. To be precise, let $\hat{\beta} = (\beta_{-t}^*, b)$. denote the strategy profile that coincides with β^* except that a consumer $t \in I_i$ deviates and selects b instead of $b^*(t)$. Then, the price system remains the same, i.e., $p(\hat{\beta}) = p^*$, and $x_t(\hat{\beta})$ is given by $x_{t,k}(\hat{\beta}) = b_k/p_k^*$. Since $\sum_{k=1}^{\infty} b_k \leq M_i = p^*\cdot\omega_i$, we have that $p^*\cdot x_t(\hat{\beta}) \leq p^*\cdot\omega_i$. This implies that $\pi_i(\beta_{-t}^*, b) = U_i(x_t(\hat{\beta})) \leq U_i(x_i^*) = \pi_i(\beta^*)$.

In spite of the fact that our approach closely follows Dubey and Geanakoplos (2003), the game we have presented could be essentially that of Shapley and

[6] See Hervés-Beloso and del Valle-Inclán (2019).

Shubik (1977). For it, consider the previous economy $\mathcal{E} = (\ell_\infty^+, \succsim_i, \omega_i)_{i=1,\dots,n}$ that, under the established assumptions, has equilibrium (x^*, p^*). From $\mathcal{E}$ and (p^*, x^*) we define the economy $\hat{\mathcal{E}}$ where agents are endowed with an amount of fiat money that is given by the value of their resources at price p^*. To be precise, for each sequence $\hat{x} = (x_0, x) \in \ell_\infty^+$, let the first coordinate $k = 0$ be a real number that represents amount of money. Thus, $\hat{\mathcal{E}} = (\ell_\infty^+, \hat{\succsim}_i, \hat{\omega}_i)_{i=1,\dots,n}$, where $\hat{\omega}_{i,0} = p^* \cdot \omega_i$, $\hat{\omega}_{i,k} = \omega_{i,k}$, for every natural number k, and where $\hat{y} = (y_0, y) \hat{\succsim}_i \hat{x} = (x_0, x)$ if and only if $y \succsim_i x$. Thus, money does not affects preferences. Analogously to the definition of $\mathcal{E}_c$, we state $\hat{\mathcal{E}}_c = (I = \bigcup_{i=1}^n I_i, \ell_\infty^+, \hat{\succsim}_t, \hat{\omega}_t)_{t\in I}$ as the n-type continuum economy associated to $\hat{\mathcal{E}}$.

Now, consider a game *à la Shapley-Shubik* but with a continuum of players $I = (0, n]$, where as before, a strategy profile $\beta = (b(t))_{t\in I}$, with $b(t) \in S_t = S_i = \{b \in \ell_1^+ \,|\, \sum_{k=1}^\infty b_k \leq p^* \cdot \omega_i\}$ if $t \in I_i$, defines a price $p_k(\beta) = \frac{\int_I b_k(t) d\mu(t)}{\sum_{i=1}^n \omega_{i,k}}$, at each trading post $k \in \mathbb{N}$, leading to the allocation that assigns to each consumer $t \in I$ the bundle $\hat{x}_t(\beta)$ as follows:

$x_{t,0}(\beta) = \omega_{i,0} - \sum_{k=1}^\infty b_k + \sum_{k=1}^\infty p_k(\beta) \cdot \omega_{i,k} \geqslant 0$, if $t \in I_i$ and

$$x_{t,k}(\beta) = \begin{cases} \frac{b_k(t)}{p_k(\beta)} \text{ if } p_k(\beta) > 0 \\ 0 \text{ otherwise} \end{cases}$$

A strategy profile $\beta^* = (b^*(t))_{t\in I}$, is a Nash equilibrium if no player has incentives to deviate individually, i.e., $\hat{x}_t(\beta^*) \hat{\succsim}_t \hat{x}_t(\beta^*_{-t}, b_t)$, for every $b_t \in S_t$ and every $t \in I$.

As before, one shows that the symmetric strategy profile β^*, with $b_k^*(t) = p_k^* \cdot x_{i,k}^*$ for each $k \in \mathbb{N}$ and $t \in I_i$, is a Nash equilibrium and results in the competitive prices p^* and allocation x^*. Moreover, we observe that, in equilibrium, every agent maintains their initial amount of money.

Therefore, given an equilibrium for an economy with infinitely many commodities, we have defined two associated market games in which the bids each agent may propose are bounded from above by the equilibrium value of the endowments. Both market games have a Nash equilibrium that results in the equilibrium of the economy, although this does not prevent other different equilibria in the game. We stress that the result holds for a finite number of goods.

To finish this section we show how the Schmeidler's game can be extended to economies with infinitely many commodites. For this, consider again the original economy $\mathcal{E} = (\ell_\infty^+, \succsim_i, \omega_i)_{i=1,\ldots,n}$. To define the associated game *à la* Schmeidler, let be S_i the strategy set of player i,

$$S_i = \{(x,p) \in \ell_\infty^+ \times \ell_\infty' \mid p \cdot x \leq p \cdot \omega_i\}.$$

For each strategy profile $s = (s_1, s_2, \ldots, s_n) = ((x_1,p_1),(x_2,p_2),\ldots,(x_n,p_n))$, let $A_i(s), \#A_i(s)$ and $\gamma_i(s)$ be defined as in Section 3. Given the profile s, each player i receives $f_i(s)$, the bundle they choose adjusted by the average excess of demand of the players that proposes the same price.

$$f_i(s) = x_i - \gamma_i(s) = x_i - \frac{\sum\limits_{j \in A_i(s)} (x_j - \omega_j)}{\#\, A_i(s)}.$$

Let (x^*, p^*) be any Walrasian equilibrium of the economy $\mathcal{E} = (\ell_\infty^+, \succsim_i, \omega_i)_{i=1,\ldots,n}$ with prices $p^* \in \ell_1$. Then, $s^* = (s_1^*, s_2^*, \ldots, s_n^*) = ((x_1^*, p^*), (x_2^*, p^*), \ldots, (x_n^*, p^*))$ is a Nash equilibrium of the game.

For it, note that for any player i and for any strategy $s_i = (x,p) \in S_i, s_i \neq {s^*}_i$, we have, either $p = p^*$ or $p \neq p^*$. If $p \neq p^*$, then $f_i(s_{-i}^*, s_i) = \omega_i$. If $p = p^*$, then $f_i(s_{-i}^*, s_i) \leqslant \frac{(n-1)x + x_i^*}{n}$. In both cases, the output $f_i(s_{-i}^*, s_i)$ is in the budget set at prices p^*, and thus, we have $f_i(s^*) \succsim_i f_i(s_{-i}^*, s_i)$.

Remarks. The game associated to the economy $\mathcal{E} = (\ell_\infty^+, \succsim_i, \omega_i)_{i=1,\ldots,n}$, following the models proposed by Shapley- Shubick and Dubey-Geanakoplos consider a continuum of players and then a change of strategy of one player does not affect the price at any trading post. In contrast, the Schmeidler's game considers as many players in the game as consumers in the economy. Note that in both, the game *à la* Shapley-Shubik and in the game by Schmeidler, any Walrasian equilibrium with prices in ℓ_1 defines an associated game with a Nash equilibrium that results in the equilibrium of the economy. However, the game following the approach by Schmeidler does not require prices to belong to ℓ_1, whereas the one derived from Dubey-Geanakoplos requires Mackey continuity of preferences to guarantee their representation by utility functions.

Shapley-Shubik's model overcomes bankruptcy problems by considering each player endowed with an initial amount of money. However, the approaches followed by Dubey-Geanakoplos and Schmeidler may have some drawbacks that, out of equilibrium, have to do with issues of default, and with non-feasibility, at an individual or aggregate level, respectively. In fact, Schmeidler (1980, page

1590) advises that one may find some profiles s for which the allocation $f_i(s)$ may be out of the positive cone, where preferences are defined, and therefore this must be considered a shortcoming. This omission is fully justified in equilibrium, but it casts doubts on the profiles that produce default as it may happen in the Dubey-Geanakoplos's model. However, the alternative would require to define a game where each agent also considers the possibility of the bankruptcy by others.

6 Remarks, some applications and future research

Manipulability of the Walrasian mechanism has been thoroughly studied by considering different scenarios and strategic considerations. In fact, it is known that full information on consumers' true endowments is not always available and obtaining such information is not easy and might be very costly. Thus, manipulation via misrepresentations of resources can be considered a quite common situation. For example, when there is excess of supply for a commodity, those who are endowed or produce it can sometimes manipulate prices to their benefit by holding or even destroying part of it. Hence, by considering misrepresentation of endowments, agents may have an incentive to deviate from a competitive behavior and manipulate prices in their own benefit.

However, these strategic considerations are not addressed in the papers we have referred to in this manuscript. Actually, in the games we have recapitulated agents put up their entire endowment for sale in the trading posts or it is implicitly considered that endowments are known and there is no strategic behavior on withholding resources. This issue is somehow remarked by the authors; Shapley and Shubik (1977) argue that it is not difficult to modify the basic game so that the goods do not necessarily all pass through the market before consumption, and a footnote in Dubey and Geanakoplos (2003) reads that the more realistic assumption that agents sell what they want would be more complicated but without affecting the result. However, a number of different considerations and problems arise depending on how this is done and, even more, an explicit analysis on the incentives that consumers may have, by withholding a portion of their endowments in order to manipulate prices in their own benefit, is required.

We remark that incentives to deviate from a price-taking behavior have been analyzed for the case in which the withheld bundles are destroyed or fully or partially available for consumption (see, for instance, Roberts and Postlewaite, 1976, and Moreno-García, 2006). Further perfect competition tests which checks the incentives of small coalitions to behave strategically have been also addressed for economies with an infinite degree of commodity differentiation (see Hervés-Beloso, Moreno-García and Páscoa, 1999). It is in our future research agenda to study strategic behavior in the market games described in this manuscript with the aim of deepening the analysis of perfectly competitive markets in contrast to market power situations.

The pioneering games stablished in the literatura in relation to market equilibria in economies have generated a variety of applications to different topics. In particular, the Shapley and Shubik model has been carried forward by several others, who took up the theme of showing that Cournot-Nash equilibria converge to Walrasian equilibria. For instance, as we have remarked, Dubey and Geanakoplos (2003), by using a variant of the Shapley-Shubik trading-post game with inside fiat money, proved existence of pure Nash equilibrium, and convergence to competitive equilibria under replication deriving also the existence of a Walrasian equilibrium. We refer the reader to Giraud (2003) for a review of the literature on strategic market games which also includes some extensions of market games à la Shapley-Shubik to financial markets.

In a couple of papers, we went further and adapted variants of the Shapley-Shubik game as developed in the above cited work by Dubey and Geanakoplos (2003) to different scenarios. Faias, Hervés-Beloso and Moreno-García (2011) provided a strategic market game approach for equilibrium price formation in markets with differentially informed agents, and Faias, Moreno-García and Wooders (2014) introduced a model of a strategic market game for the private provision of public goods and related the equilibria of the game with the private-provision equilibrium which is a counter-part to the Walrasian equilibrium for an economy with multiple private and public goods.

As an extension and application of Schmeidler's (1980) market game, Fugarolas et. al (2009) recasted a differential information economy as a strategic game in which players propose net trades and prices. For it, they proposed a market game mechanism that links Schmeidler type outcome functions and a delegation rule, as well as it allows agents to inform anonymous players about their objective functions (who, by themselves, incorporate the information constraints). Their main result shows that pure strategy Nash equilibria are strong and determine both consumption plans and commodity prices that coincide with the Walrasian expectations equilibria of the underlying economy.

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