ECOBAS Working Papers
2019 - 04

Title:

STRICTLY MONOTONIC PREFERENCES

Authors:

Carlos Hervés-Beloso
Universidade de Vigo

Paulo K. Monteiro
Universidade de Vigo

Economics and Business Administration for Society
www.ecobas.es

Strictly monotonic preferences

Carlos Hervés-Beloso
ECOBAS. Universidad de Vigo.
e-mail: cherves@uvigo.es

Paulo K. Monteiro
FGV EPGE Rio de Janeiro.
e-mail: Paulo.Klinger@fgv.br

Abstract. Monotonicity assumptions of preferences are natural and useful. A strictly monotonic preference is such that an increase in even only one commodity consumption is always strictly preferred. However, when we consider a continuum of commodities, it is not easy to find examples of strictly monotonic preferences. We survey some previous results in order to show that purely strictly monotonic preferences always exist but, if the commodity space is rich enough, they cannot be continuous in any linear topology defined on the consumption set and they cannot be represented by a utility function.

Keywords: Strictly monotonic preferences, pure monotonicity, continuum of commodities, continuity of preferences, utility representation.

JEL Classification: D11, D50

*This work is partially supported by Research Grants ECO2016-75712-P (Ministerio de Economía y Competitividad) and ECOBAS (Xunta de Galicia). Monteiro acknowledge the financial support of CNPq–Brazil

1 Introduction

The preference relation of a consumer or a decision maker is strictly monotonic if an increase in even only one commodity consumption is always strictly preferred. This work addresses existence, continuity and utility representation of such preferences. The existence of a utility function representing a preference is closely related with the properties of the set of alternatives to which we often refer as the consumption set. As we focus on continuous preferences, the topology of the consumption set will play a relevant role.

In case of finite or countable many commodities, it is easy to find examples of continuous and strictly monotonic preferences. In this scenario, the existence of utility functions is guaranteed by Eilenberg-Debreu Theorem, as soon as the underlying topological space is connected and separable or second countable. Yet, it is not an easy task to give examples of strictly monotonic preferences defined on general commodity spaces involving uncountable many commodities. Moreover, in this situation, a representative commodity space, as $B([0,1])$, the space of bounded functions defined on $[0,1]$, fails to be separable and thus, the classical utility representation results do not apply.

Several authors have elaborated on generalizations of the Eilenberg-Debreu Theorem providing conditions for the existence of utility representation. We refer to Fleischer 1(961), Nachbin (1965), Peleg (1970), Jaffray (1975), Mehta (1977,1988), Richter (1980), Herden (1989). See Bridges and Mehta (1995) for a complete summary of this contributions and Stigler (1950) for a historical analysis of utility theory.

However, as Estévez and Hervés (1995) have shown, in every non-separable metric space, there are continuous preferences that can not be representable by a utility function. Yet, these non-representable preferences, are neither convex nor monotonic.

In Section 2.1 we set the notations and we provide the background. The positive and negative results on preference representation are presented in Section 2.2 . Since our objective focuses on the case of uncountably many commodities, we survey the findings by Monteiro (1987) and the generalization provided by Candeal *et al.* (1999) which give necessary and sufficient conditions for a continuous preference defined on a non separable topological space to be representable by a utility function.

In Section 3, we provide examples of strictly monotonic and continuous utility functions defined on particular subspaces of the space of functions defined on the set of commodities $K = [0, 1]$. Moreover we stress that, for any set of commodities K, and any consumption set, there are strictly monotonic preferences.

Finally, in Section 4, we state our main result. Strictly monotonic preferences on the positive cone of spaces like the Banach space of all bounded function defined in $[0, 1]$ are neither representable by utility functions nor continuous in any linear topology. We revisit Hervés-Monteiro (2010) to extend this result to *rich-enough* consumption subsets of the space of functions defined on any non-countable set of commodities.

2 Notation, definitions and background

2.1 Definitions and notation

Let X be a set. A subset $R \subset X \times X$ is a binary relation. We write xRy if $(x, y) \in R$.

Definition 2.1 *The binary relation, R on X is:*

a) *reflexive if xRx for all $x \in X$;*

b) *transitive if xRy and yRz implies xRz;*

c) *antisymmetric if xRy and yRx implies $x = y$;*

d) *complete (or total) if for every x, y in X either xRy or yRx;*

e) *a partial order if is reflexive, transitive and antisymmetric.*

An order is a complete partial order.

Definition 2.2 *The ordered set $(K, \geqslant)$ is well ordered if for every non–empty subset $A \subset K$ there is $a \in A$ such that $x \geqslant a$ for every $x \in A$.*

That every set can be well ordered is Zermelo's theorem.

A reflexive and transitive binary relation $\succsim$ on X is a preference relation on X. If $(f, g) \in \succsim$, we write $f \succsim g$. Thus:

Reflexivity $f \succsim f$ for all $f \in X$;

Transitivity If $f \succsim g$ and $g \succsim h$, then $f \succsim h$.

We read $f \succsim g$ as f is at least as preferred as g. Moreover, for $f, g \in X$ we write $f \succ g$, and we read f is more preferred than g, if $f \succsim g$ but $g \not\succsim f$ and $f \sim g$, and we read f is indifferent to g, if $f \succsim g$ and $g \succsim f$.

A preference $\succsim$ defined on a convex X is said to be convex if, for any two points f, $g \in X$ and $\lambda \in (0,1)$, $f \succsim g$ implies $f \succsim \lambda x + (1-\lambda)g$, and strictly convex if $f \succsim g$ and $\lambda \in (0,1)$ imply $f \succ \lambda f + (1-\lambda)g$.

Let $(X, \geqslant)$ be a partially ordered set. A preference $\succsim$ defined on X, is monotonic if, for $f, g \in X$, $f \geqslant g$ implies $f \succsim g$. The preference $\succsim$ is strictly monotonic if $f \geqslant g, f \neq g$, implies $f \succ g$.

The topological space (X, τ) is said separable if it contains a countable subset whose closure is X. That is, there exists a sequence $Q = \{q_1, \ldots, q_n, \ldots\}$ such that $Q \cap V \neq \emptyset$ for every non-empty open set $V \subset X$. The topological space (X, τ) is second countable or perfectly separable if τ admits a countable basis of open sets. Every perfectly separable topological space is separable, and every separable metric space is perfectly separable. A topological space (X, τ) is connected if there is no partition of X into two disjoint, non-empty closed sets. Also, X is path-connected if for all f, $g \in X$ there exists a continuous function $f : [0,1] \to X$ with $f(0) = f$ and $f(1) = g$. Note that every path-connected space is connected and every convex set in a linear topological space is path-connected.

A preference $\succsim$ defined on a topological space (X, τ) is continuous if, for all $f \in X$, the sets $L_f = \{g \in X : f \succsim g\}$ and $U_f = \{g \in X : g \succsim f\}$ are τ-closed in X and the sets $\mathring{L}_f = \{g \in X : f \succ g\}$ and $\mathring{U}_f = \{g \in X : g \succ f\}$ are open sets for all $f \in X$. Note that if $\succsim$ is complete, then, for all $f \in X$, the sets U_f and L_f are closed if and only if the sets $\mathring{U}_f$ and $\mathring{L}_f$ are open. However, for example, in the Pareto order in $\mathbb{R}^n$, given by $a \succsim_P b$ if and only if $a_i \geqslant b_i$ for all $i = 1, \cdots, n$ we have that U_a and L_a are closed sets for all $a \in \mathbb{R}^n$, but $\succsim_P$ is not continuous since neither $\mathring{U}_a$ nor $\mathring{L}_a$ are open sets.

The following theorem shows that continuous preferences on convex sets are automatically complete:

Theorem 2.1 (Schmeidler, 1971) *A continuous preference, $\succsim$, defined on a connected space X, and non trivial (there are f and $g \in X$, such that $f \succ g$), is* *complete.*

Thus, in order to consider non-trivial continuous preferences, i. e., reflexive, transitive and continuous binary relations, we must assume that they are complete.

If $U : X \to \mathbb{R}$ is a function then $\succsim_U := \{(f,g) \in X^2; U(f) \geq U(g)\}$, that is, $f \succsim_U g$ if and only if $U(f) \geqslant U(g)$, is a complete preference relation on X. Also $f \succ_U g$ if and only if $U(f) > U(g)$. A preference relation is representable by a utility function if there is a $U = U_{\succeq} : X \to \mathbb{R}$ such that $f \succsim g$ if and only if $U(f) \geq U(g)$.

2.2 Existence of utility representation

The representability of a preference relation by a utility function was taken first as a cardinal concept to measure a consumer's well-being by Pareto (1896). However, its relevance was recognized much later by Slutsky (1915) and especially by Wold (1943), who listed a number of axioms (or conditions) that preference must meet in order to guarantee the existence of a real-valued utility representation.

One basic requirement of a utility function in applications to consumer theory is that the utility function be continuous. For continuous preferences, the positive results are very general. The **Eilenberg-Debreu Theorem**, establish that a continuous preference defined on a connected and separable, or second countable topological space (X, τ), has a continuous utility representation.

Theorem 2.2 (DEBREU, 1954)

a) *Let X be connected and separable. Then any continuous preference relation on X has a utility representation.*

b) *Let X be a perfectly separable space. Then any continuous preference relation on X has a utility representation.*

The Debreu's (1954), (see also Debreu 1964), contribution on the existence of continuous utility representation of preferences is based on an earlier work by Eilenberg (1941), that shows the existence of utility for a continuous total order in connected and separable spaces. For connected and separable spaces Debreu extended to preference relations the Eilenberg's result for total orders. Without requiring connectedness, but strengthening separability, Debreu's result shows existence of a continuous utility for continuous preferences defined on second countable topological spaces.

Note that for any utility function U representing a preference $\succsim$, if ψ is strictly increasing function $\psi : \mathbb{R} \to \mathbb{R}$, the function $V = \psi \circ U$ is also a utility function for $\succsim$. Thus, a continuous preference can be represented by a utility function V that fails to be continuous but, using the gap theorem (Debreu, 1964), it is shown the existence of a continuous function U representing the same preference.

Many authors elaborated on the problem of preference representation of preferences. We refer to Bridges and Mehta (1995), for a survey of different approaches and also to Herden (1995) for the connections between these approaches.

The most commonly known non representable preference is the lexicographic order $\succsim_L$ defined, for example, in $X = [0,1]\times[0,1]$ which is, in fact, a total order. Note that $\succsim_L$ is not continuous when one considers the Euclidean topology on X. However, it is continuous in the order space $(X, \succsim_L)$, but this topological space is non- separable.

Thus, connectedness and separability or second countability guarantee the existence of a utility representation of a continuous preference. Note that second countability is a hereditary property that, in particular, implies that any continuous preference defined on an arbitrary subset of any separable metric space is representable. Yet, in some interesting economic applications the set of alternatives could be non-separable. For example, in a situation where alternatives are renewable natural resources in continuous time or with an infinite temporal horizon, the decision-maker deals with infinitely many commodities in the commodity space L_∞ or l_∞ (see Bewley 1972 , Araujo 1985). In the case where one consider an infinite degree of commodity differentiation, the space of alternatives, following Mas-Colell (1975), is $ca(K)$, the space of countably additive signed measures over the compact space of commodities K. The respective positive cones of the Banach spaces $ca(K)$, L_∞, l_∞, that usually play the role of consumption set, are non-separable metric spaces.

Moreover, given any non-separable metric space (X, d), Estévez and Hervés (1995) show that there are continuous preference relations defined on X which cannot be represented by a utility function. To show this general non-existence result, it is used the characterizing property of non-separable metric spaces. In any non-separable metric space (X, d) there is an uncountable set $I \subset X$ and a real number $\epsilon > 0$ such that for every two different points f, $g \in I$, $d(f,g) > 2\epsilon$. They define a preference $\succsim$ such that each point $f \in I$ is the most desirable point in a ϵ neighborhood of f, and for any point g, outside the ϵ neighborhood of any point $f' \in I$ we have $f' \succ g$. By using the *long line*[1], it is shown that $\succsim$ is non-representable by any utility function.

Yet, the class of non-representable preferences that guarantees the above result does not fulfill neither monotonicity, if X is a partially ordered metric space, nor convexity, when X is a convex set, nor local insatiability.[2] However, Monteiro (1987) provides an example[3] of a convex, monotone and continuous preference defined on a closed convex subset of a Banach lattice that has no utility representation.

Following Monteiro (1987), given a preference $\succsim$ defined on X, a subset $F \subset X$ bounds $\succsim$ if for any $f \in X$, there are points $q_f, q^f \in F$ such that $q_f \succsim f \succsim q^f$. We say that $\succsim$ is countably bounded if there is a finite or countable set $F \subset X$ that bounds $\succsim$. It is easy to show that countable boundedness is a necessary condition for a preference to be representable by a utility function. Moreover, Monteiro (1987) showed that it is also sufficient to guarantee the existence of a continuous utility representation for continuous preferences defined on the usual consumption sets. More precisely it is shown that:

If (X, τ) is path connected, any countably bounded continuous preference relation $\succsim$ has a continuous utility function (Theorem 3, page 150).

In fact, this result is based on Theorem 1 in Monteiro (1987) that shows that if there is $F \subset X$ connected and separable which bounds $\succsim$, then $\succsim$ has a continuous utility representation. Following this idea, Candeal *et al.* (1998) consider the following definition:

A topological space (X, τ) is separably connected if and only if, given any two points $f, g \in X$ there is a connected and separable set, $F_{f,g} \subset X$ such that $f, g \in F_{f,g}$.

A separably connected space is connected and a path-connected topological space is separably connected because every path is connected and separable. However, not every separably connected space is path-connected.

[1]See Steen and Seebach,1970, pp. 71–72

[2]The preference $\succsim$ is locally insatiable if, for any point g in the consumption set X, and for any neighborhood V of f, there is another consumption $f \in V$ such that $f \succ g$.

[3]It is theorem 6 on page 153

Theorem 4 in Candeal *et al.* (1998), shows that a continuous preference defined on a separably connected topological space is representable by a continuous utility function if and only if it is countably bounded.

For examples of separably connected topological spaces that are neither path-connected nor separable we refer to Candeal *et al.* (1998). Actually, it was conjectured that any connected subset of a linear or metric space is separably connected but Aron and Maestre (2003) provide a counterexample to this conjecture. See also Wójcik (2016).

On the other hand, if X is totally ordered by $\geqslant$ and we consider the order topology, then $(X, \geqslant)$ is separably connected if and only if it is path-connected.

Consider the lexicographic order $\succsim_L$ defined in $\mathbb{R}^n$. Note that $\succsim_L$ is countably bounded and if we consider the order topology in $\mathbb{R}^n$, $\succsim_L$ is a continuous preference. However the ordered space $(\mathbb{R}^n, \succsim_L)$ is not path-connected.

3 Strictly monotonic preferences

The property of monotonicity of preferences represents the idea that more is better. A preference $\succsim$ defined on a partially ordered set $(X, \geqslant)$ is monotonic if $f \geqslant g$ implies $f \succsim g$ and it is strictly monotonic if $f \geqslant g$ and $f \neq g$ implies $f \succ g$. In consumer theory, preferences are defined on the consumption set X. An element $f \in X$ represents a consumption plan that specifies the units, $f(k)$, of each commodity k that the consumer chooses to consume. Without loss of generality we assume that K is the set of commodities and $X \subset \{f : K \to \mathbb{R}_+\}$ is a set of non-negative functions defined on K. Thus, we can consider the standard partial order on X, that is $f \geqslant g$ if and only if $f(x) \geqslant g(x)$ for all $x \in K$. The following is theorem 2 in Hervés-Monteiro (2010):

Theorem 3.1 *Given a set K and any subset $X \subset \{f : K \to \mathbb{R}\}$, there are strictly monotonic preferences defined on X.*

Given any set K, let $\geqslant^*$ be a well-ordering of K. Let $f, g \in X, f \neq g$ and let $k_o = \min\{k \in K; f(k) \neq g(k)\}$. If $f(k_o) > g(k_o)$ we define $f \succ_L g$ and otherwise, we define $g \succ_L f$. Note that $f, g \in X, f \geqslant g, f \neq g$ implies $f \succ_L g$. The relation $\succsim_L$ defined by $f \succsim_L g$ if $f \succ_L g$ or $f = g$ is a strictly monotonic preference (in fact it is a strictly monotonic total order) on X.

When K is a finite set with n points we have that $X \subset \mathbb{R}^n$ and in case where K is a countable set, X is a subset of the space of sequences of real numbers. Note that $\succsim_L$ is the restriction of the lexicographic order in $\mathbb{R}^n$, respectively $\mathbb{R}^{\mathbb{N}}$ to X, which is neither continuous nor representable by a utility function.

It is very easy to find examples of continuous strictly monotonic preferences, representable by utility functions. In the countable case, the Banach spaces of bounded sequences, l_∞, and absolutely summable sequences, l_1, have been used to represent, respectively, economies with renewable and non-renewable resources. Consider $X \subset l_\infty^+$; given any sequence $\rho = (\rho_n)_{n \in \mathbb{N}}$, where $\rho_n > 0$ for all n, and $\sum_{n=1}^{\infty} \rho_n < +\infty$, the preference relation defined in X by $f \succsim_\rho g$ if and only if $u_\rho(f) \geqslant u_\rho(g)$, where $u_\rho(h) = \sum_{n=1}^{\infty} \rho_n h(n)$, is strictly monotone, continuous in both the norm and the weak* topology $\sigma(l_\infty, l_1)$, and, by definition, representable by the utility function u_ρ. Similarly, if the consumption set $X \subset l_1^+$, given any bounded sequence $b = (b_n)_{n \in \mathbb{N}}$, the preference relation defined in X by $f \succsim_b g$ if and only if $u_b(f) \geqslant u_b(g)$, where $u_b(h) = \sum_{n=1}^{\infty} b_n h(n)$, defines a continuous and strictly monotonic preference on X.

When K is an uncountable set, for instance $K = [0, 1]$, the commodity spaces like $C(K)$, the space of continuous functions defined on the compact set K, $L_p(K)$, with $1 \leqslant p < \infty$, the spaces of classes of functions for which the p-th power of the absolute value is Lebesgue integrable, $L_\infty(K)$, the space of classes of essentially bounded measurable functions defined on K, or the space $B(K)$ of bounded functions defined on K have been considered in the literature. For consumption sets contained in the positive cone of the spaces $C(K)$ or $L_p(K)$, it is easy to find examples of strictly monotonic preferences, representable by utilities. For example, $U(f) = \int f$, or respectively, $U(f) = \int f^p$ defines a strictly monotonic preference in $C(K)^+$ and $L_\infty(K)^+$, respectively in $L_p(K)^+$.

Moreover, we observe that given two continuous functions f, g defined, for example, in $K = [0, 1]$, such that $f \geqslant g$ and $f \neq g$, we have $f(k) > g(k)$ for uncountably many points $k \in K$. The same happens if f, g represent classes of integrable functions. Thus, when K is uncountable, the natural order in $C(K)$ or in the spaces $L_p(K)$, with $1 \leqslant p \leqslant \infty$, fails to represent, in a proper way, the idea that to consume more than just one commodity.

For this reason we focus on strictly monotonic preferences defined in $X \subset B(K)$ or, with more generality, $X \subset F(K) = \{f : K \to \mathbb{R}\}$. Note that any continuous and monotonic preference defined in the linear space of bounded functions $B(K)$ or in the positive cone $B(K)^+$ is countably bounded (by the set of constant functions with rational values) and, consequently, it has a continuous utility representation. However, contrary to the case of $C(K^+)$ or $L_p(K)^+$, it is not so easy to find examples of strictly monotonic preferences in the positive cone of the space of all bounded functions $B(K)$.

Next section elaborates on this difficulty.

4 Strictly monotonic preferences on $F(K)$

Along this section K denotes an uncountable set and we consider first the linear space $F([0,1])$ of all functions defined on $K = [0,1]$. Our main result is the following

Theorem 4.1 (Non–existence) *Every strictly monotonic preference relation on X, the positive cone of the space functions defined on $[0,1]$, is non-representable.*

Proof. Let $\succsim$ be any strictly monotonic preference relation defined on X. Suppose $U : X \to \mathbb{R}$ represents $\succsim$. Define for $t \in [0,1]$ the functions $g_t = \chi_{[0,t)}$ and $f_t = \chi_{[0,t]}$, where χ_A to denote the characteristic function of the set A; that is, $\chi_A(x) = 1$ if $x \in A$ and is 0 otherwise. Since $f_t > g_t$ we have that $U(f_t) > U(g_t)$. Now if $s > t$ we have that $g_s > f_t$ and therefore $U(g_s) > U(f_t)$. In particular we conclude that the family of intervals $\{I_t : t \in [0,1]\}$, where $I_t = (U(g_t), U(f_t))$, is pairwise disjoint and uncountable. This impossibility shows the result.

The Corollary is theorem 3 in Hervés-Monteiro (2010):

Corollary 4.1 (Non–continuity) *Let τ be a linear topology on X. Then, no strictly monotonic preference on X can be continuous.*

Proof. Let $\succsim$ be any strictly monotonic preference relation defined on X, and let us consider the restriction of $\succsim$ to X_B, the subset of non-negative bounded functions. As we already have observed, monotonicity implies that $\succsim$ is countably bounded in X_B. If $\succsim$ were continuous, since X_B is convex, it would have a (continuous) utility representation, which is forbidden by Theorem 4.1.

Remark: Note that the proof of Theorem 4.1 only requires that the order interval $[0, \chi_{[0,1]}] \subset X$, that is a convex set. Thus, the theorem and the corollary apply for any set X containing the order interval $[0, \chi_{[0,1]}]$

The negative results stated above refer to the standard consumption set of the commodity spaces $F([0,1])$ or $B([0,1])$ and, obviously, cannot be extended to any subset X, as the previous examples ($X \subset \mathbb{R}^n, l_\infty, l_1$ C[0,1], $L_p[0,1]$, with $1 \leqslant p \leqslant \infty$) show.

Finally, we will replicate Theorem 4.1 and Corollary 4.1 in the case where K is any uncountable set and $X \subset F(K)$ is a "rich enough" set of consumption plans.

Let be two consumption plans $f, g \in X, f \geqslant g$. It is natural to assume that if a consumer (a decision maker) can choose both f and g, she also can choose any consumption plan h with $f \geqslant h \geqslant g$. That means that if a consumer can consume an amount $f(k)$ or $g(k)$ of commodity k, she also can consume any intermediate value. Note that this property is neither fulfilled when X is a set of continuous functions nor when X is a set of classes of integrable functions. Note also that to consider continuous functions on K implies a close relation among commodities; if the consumer chooses an amount $f(k)$, she is obliged to choose a similar amount of every commodity $k^{'}$ in a neighborhood of k.

In order to show next results, we will require that the consumption set X be *"rich enough"*. A set $X \subset F(K)$ is said *"rich enough"* if there are two functions $f, g \in X$ such that $f \geqslant g, f(k) > g(k)$ for all $k \in K_o \subset K$ with K_o uncountable, and such that the segment $[g, f] = \{h; g \leqslant h \leqslant f\} \subset X$.

Theorem N' Let K any uncountable set and let be $X \subset F(K)$ *"rich enough"*. Every strictly monotonic preference relation on X is non-representable.

The proof is, essentially, the same as in Theorem 1 in Hervés and Monteiro, (2010). See also Hervés and del Valle-Inclán Cruces (2019).

Let $\geqslant_K$ be a total order in K, the assumption guarantees that there are g and $f \geqslant g$ [4] functions on K such that $[g, f] \subset X$ and $g(k) < f(k)$ for all k in the uncountable set K_o. For any $k \in K_o$, define $h_k(x) = f(x)$ if $k >_K x$ and $h_k(x) = g(k)$ otherwise, and $f^k(x) = f(x)$ if $k \geqslant_K x$ and $f^k(x) = g(x)$ otherwise. Note that $h^k > h_k$ for all $k \in K_o$ and thus, if U is a utility representing a strictly monotonic preference $\succsim$, for each $k \in K_o$, we would have that $I_k = (U(h_k), U(h^k))$ would be a non-empty open interval and for any other point $k' >_K k, h_{k'} > h^k$ and thus, $I_k \cap I_{k'} = \emptyset$. We would have uncountably many non-

[4]Note that $\geqslant_K$ is a total order on K, whereas $\geqslant$ represent the natural partial order in the set of functions X

empty open and disjoint real intervals, an impossibility that shows the result.

Corollary (N-C)'

Let τ be a linear topology[5] on $X \subset F(K)$ *"rich enough"*. Then, no strictly monotonic preference on X can be continuous.

Proof. Let $\succsim$ be any strictly monotonic preference relation defined on X, and let g, f as in the previous proof. Let us consider the restriction of $\succsim$ to the set $X' = [g, f] \subset X$. Note that $\succsim$ is bounded in the convex set X' by $\{g, f\}$. If $\succsim$ were continuous it would have a (continuous) utility representation in X' that is *"rich enough"*, but this contradicts Theorem N'.

Remark: A relevant example of a *"rich enough"* consumption set is $B(K)^+$, the positive cone of $B(K)$, the Banach space of all bounded functions defined on K. We have shown that there are strictly monotonic preferences defined on $B(K)^+$, but none of them is continuous and none of them has a utility representation. Yet, there are infinite dimensional subspaces of $B(K)$ for which there are continuous and strictly monotonic preferences with utility representation defined on the positive cone.

For it, consider $\ell^1([0,1]) \subset B(K)$. A bounded function $f \in \ell^1_+([0,1])$ is such that $\{t; f(t) \neq 0\}$ is countable and $\|f\|_1 = \sum_{t\in[0,1]} f(t) < \infty$. Define $f \succsim g$ if and only if $U(f) \geqslant U(g)$. Observe that $\succsim$ is strictly monotonic, continuous, with the topology given by the norm $\|\|_1$ and representable by the utility U. This does not contradict the previous results since $\ell^1_+([0,1])$ is not *"rich enough"*. In this consumption set no agent ever consumes an uncountable set of commodities.

[5]Linearity guarantee that the path $t \to (1-t)a + tb$ joining a to b is continuous.

References

Araujo, A. P. (1985), Lack of Pareto Optimal Allocations in Economies with Infinitely Many Commodities: The Need for Impatience, Econometrica, v. 53 (2), pp. 455-461

Aron, R.M., Maestre, M. (2003): A connected metric space that is not separably connected. Contemporary Mathematics 328: 39–42.

Bewley, T. (1972),Existence of equilibria with infinitely many commodities, Journal of Economic Theory v. 4, 514-540.

Bridges, D., Mehta, G.: Representations of Preferences Orderings, Lecture Notes in Economics and Mathematical Systems 422, Springer, Berlin, 1995

Candeal, J.C., Hervés-Beloso, C., Induráin, E. (1998): Some results on representation and extension of preferences. Journal of Mathematical Economics 29(1): 75–81.

Debreu, G., 1954, Representation of a preference ordering by a numerical function, in: R.M. Thrall, C.H. Coombs and R.L. Davis, eds., Decision Processes (Wiley, New York) 159–165; also in Mathematical economics: twenty papers of Gerard Debreu (Cambridge University Press, Cambridge), 105–110. Eilenberg, S. (1941), Ordered Topological Spaces, American Journal of Mathematics, Vol. 63, No. 1 (Jan., 1941), pp. 39-45

Estévez, M. and Hervés, C., (1995): On the existence of continuous preference orderings without utility representations, Journal of Mathematical Economics 24, 305–309.

Fleischer, I., (1961): Numerical representation of utility. J. Soc. Ind. Appl. Math. 9(1), 48–50

Herden, G., (1989): On the existence of utility functions. Math. Soc. Sci. 17, 297–313

Herden, G., 1989): On the existence of utility functions II. Math. Soc. Sci. 18, 107–117

Hervés-Beloso, C., Monteiro, P.K., (2010) Strictly monotonic preferences on continuum of goods commodity spaces. Journal of Mathematical Economics 46(5): 725–727

Hervés-Beloso, C., del Valle-Inclán Cruces, H., (2019) Continuous preference orderings representable by utility functions. Journal of Economic Surveys 33, 1, 179–194

Jaffray, J.-Y., (1975): Existence of a continuous utility function: an elementary proof. Econometrica 43, 981–983

Kelley, J.: General Topology, 1955, Graduate Texts in Mathematics 27

Mas Colell, A., (1986). The price equilibrium existence problem in topological vector lattices. Econometrica 54 (5), 1039–1054.

Mehta, G.,(1977): Topological ordered spaces and utility functions. Int. Econ. Rev. 18(3), 779–782

Mehta, G.,(1988): Some general theorems on the existence of order preserving functions. Math. Soc. Sci. 15, 135–143

Mehta, G.B. (1998) Preference and utility. In S. Barber'a, P.J. Hammond and C. Seidl (eds.), Handbook of Utility Theory, Chapter 1 (pp. 1–47). Boston: Kluwer Academic Publishers.

Monteiro, P.K., 1987. Some results on the existence of utility functions on path connected spaces. Journal of Mathematical Economics, 147–156.

Nachbin, L.: Topology and Order, Princeton. D. Van Nostrand Company, New Jersey (1965)

Pareto, V. (1986) Course d' economie politique. Lausanne: Rouge. See also Manuale di economia politica. Milan: Societa Editrice Libraria (1906). Reprinted as Manual of Political Economy. New York: Augustus M. Kelley, 1971.

Peleg, B.: Utility functions for partially ordered topological spaces. Econometrica 38, 93–96 (1970)

Richter, M.: Continuous and semi-continuous utility. Int. Econ. Rev. 21(2), 293–299 (1980)

Schmeidler, D.: A condition for the completeness of partial preference relations. Econometrica 39(2), 403–404 (1971)

Slutsky, E. (1915) Sulla Teoria del Bilancio del Consumatore. Giornale Degli Economisti 51(1): 1–26.

Steen, L.A. and Seebach, J.A.J. (1970) Counterexamples in Topology. New York: Holt, Rinehart and Winston Inc.

Stigler, G.J.: The development of utility theory. I. J. Polit. Econ. 58(4), 307–327 (1950)

Stigler, G.J.: The development of utility theory. II. J. Polit. Econ. 58(5), 373–396 (1950)

Wójcik (2016), The Generalized Aron–Maestre Comb, Houston Journal of Mathematics v. 42 (2), 701–707