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Offer Curves
and
Uniqueness of Competitive Equilibrium*

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Abstract

We establish sufficient conditions to guarantee uniqueness of the competitive equilibrium in a two-commodity, two-agent exchange economy. If agents' offer curves share a common directional monotonicity property –i.e., at least one commodity is always normal for all agents–, then competitive equilibrium is unique. If not, we can provide testable conditions for uniqueness by restricting the support of the distributions of individuals' preferences and endowments that characterize the agents' offer curves. The conditions are checked in well-known utility representations of preferences commonly used in the literature to illustrate the failure of uniqueness of equilibrium.

Keywords: offer curve, competitive equilibrium, uniqueness.
JEL: D52, D61, G12

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"The uniqueness of general equilibrium price is a central topic in economic theory, and without it conclusions of comparative statics and dynamics may degenerate to null statements."
Kamiya (1984, p.81)

1 Introduction

The potential existence of several relative price ratios that coordinate decentralized individual self-motivated decisions is one of the most disappointing results in general equilibrium theory. A wide range of areas in economic theory share this concern, such as international trade –an issue of interest already found in Marshall (1879, Chap.I) (see also Chipman 1965, Sec.2.6)–, monetary economics (Cass et al. 1979), endogenous business cycles (Grandmont 1985), or applied general equilibrium models (Kehoe 1998). As is well-known, uniqueness of the Walrasian competitive equilibrium cannot be guaranteed by simply imposing standard assumptions on economic fundamentals (e.g., those on individual agents' consumption set, preferences, endowments or technology). To assure that equilibrium is globally unique in an exchange economy, a number of conditions have been assumed in the equilibrium manifold[1] and in the excess demand function.[2] Concerning the former, a unique competitive equilibrium exists by restricting the topological properties of the equilibrium manifold[3] or assuming particular features of the equilibrium prices for every economy.[4] Concerning the latter, at most one (normalized) price equilibrium exists if one of the following conditions is satisfied: (i) the aggregate excess demand function satisfies either the weak axiom of revealed preference or the gross substitute property;[5] (ii) the economy has diagonal dominance;[6] or, (iii) the Jacobian of the excess demand function is negative semidefinite.[7] Also, as a device to test uniqueness, the Index Theorem can be applied to study the sign of the determinant of the (normalized) Jacobian of the excess demand function evaluated at an equilibrium price.[8] Potent as these conditions are for establishing uniqueness, they all share a basic flaw: the optimizing behavior of individual agents does not suffice to guarantee that these properties hold both for the equilibrium manifold or the aggregate excess demand function. Thus, "it is of considerable interest to try to develop conditions on the primitives of the economy for [uniqueness] to hold." (Bryant 2010, p.182).

[1] The *equilibrium manifold* is defined as the set of pairs of prices and redistribution of the aggregate endowment such that the aggregate excess demand function is equal to zero (see Debreu 1970 or Balasko 1975).

[2] See different surveys such as Arrow et al. (1971, Chap.9), Kehoe (1985, 1998), Mas-Colell (1991), Mas-Colell et al. (1995, Chap.17F), Mukherji (1997), and Bryant (2010, Chap.9).

[3] Specifically, three topological properties guarantee uniqueness: (i) the Pareto allocations belong to a (feasible) connected component of the union of connected components of regular values of the Debreu mapping (Balasko 1975, Th.4); (ii) the equilibrium manifold exhibits zero curvature (Loi et al. 2018a, Th.5.1); or, (iii) the normal vector field of the equilibrium manifold is constant outside a compact subset of the ambient space (Loi et al. 2018b, Ths.3.1 and 4.1).

[4] Namely, the independence of the equilibrium prices on the aggregate endowment for any redistribution of aggregate endowment (Balasko 1988, Theorem 7.3.9(2)).

[5] Mas-Colell et al. (1995) Propositions 17.F.2 and 17.F.3, respectively.

[6] See Arrow et al. (1971, Th. 9.12). This refers to the 'higher sensibility' of the excess demand for each good to a change in the price of that good than to a change in the prices of all other goods combined, together with the assumption that one good is always desirable.

[7] Arrow et al. (1971, Ths. 9.1, 9.2 and 9.13). That is, the Jacobian of the excess supply function –the negative of the excess demand function– whether it satisfies the Gale-Nikaidô property or it is positive defined. See also Pearce et al. (1973, 1974).

[8] Kehoe (1998, Secs.3.2-3.4).

In this paper, we provide sufficient conditions for uniqueness of the competitive equilibrium by restricting the (discrete) distribution of individuals' preferences and endowments in an exchange economy with two commodities and two agents. More specifically, our analysis shows that constraining the shape and properties of offer curves –i.e., the projection of the each individual's demand function on the consumption set– has relevant consequences on the existence of a unique (normalized) price vector.

In a setting similar to our exchange economy, but populated by a continuum of consumers in which a continuous, multidimensional distribution of individual characteristics generates the (mean) aggregate excess demand function, Hildenbrand (1983) initiated a line of research aiming to find conditions guaranteeing uniqueness.[9] The conditions proposed by this literature restrict the heterogeneity of the distribution of individual's preferences and/or endowments to satisfy particular aggregate regularities, so that the (mean) excess demand function fulfills one of the aforementioned conditions for uniqueness.[10] Note that the results in this approach depend crucially on the assumption of a continuum of consumers (Kirman et al. 1986). Hence, these results need not be satisfied in exchange economies populated by a discrete number of types of individuals, as is the case in our setting. For instance, in Section 6 we provide examples showing that the assumption of a collinear distribution of endowments, as stated by Hildenbrand (1983, Prop.1), is not enough for uniqueness.

Unlike this line of research, our work focuses on the properties of the agents' offer curve to address uniqueness.[11] The appealing of our approach is that the intersection of offer curves has traditionally been used in different areas of economic theory to graphically and intuitively identify competitive equilibria at the Edgeworth box. Two articles in the literature have followed our approach in the same setting. Balasko (1995, Th.3) provided a general condition on the individual's offer curves to guarantee a unique competitive equilibrium. However, Balasko's condition is a topological one, thus reducing the extent of its applicability. In contrast, we explore how the shapes of offer curves relate to the agents' primitives; this allows us to propose testable conditions. Chipman (2009) finds a condition for global stability (and hence, for uniqueness) of competitive equilibrium by restricting the slope of the offer curves in a particular non-tâtonement dynamic-adjustment process. Yet, their results are restricted to a CES economy under a mirror supersymmetry assumption, which imposes that the agents' preferences and endowments are mirror images of one another. Our aim is to provide more general conditions for individual fundamentals.

In this paper, we focus on an exchange economy with two commodities and two agents –two groups of homogeneous agents or two countries– to highlight the idea that restricting the properties of the aggregate excess demand function is not the only way to guarantee uniqueness of the competitive equilibrium. To be more precise, we provide sufficient conditions for uniqueness that restricts the way in which the net demand function of an agent (or one side of the market) behaves relative to the behavior of the net supply function of the other agent (or the other side of the market). We show that conditions on *individual* offer curves displaying a particular monotonicity

[9]For a survey see Hildenbrand (1999).

[10]The restrictions account for: (i) constraining the shape of the density of price-independent income distribution (Marhuenta 1995, Th.5); or, (ii) requiring that the mean income effect matrix, which belongs to the aggregation of the Slutsky decomposition across all agents, satisfies the the *increasing spread* property (i.e. it is positive-semidefinite) –see Jerison et al. (2008) for a comprehensive survey of the conditions that satisfy this property, such as those provided by Hildenbrand (1983, Prop.1), Grandmont (1992, Th.3.3), Quah (1997, Th.5.2) and Jerison (1999, Prop.1).

[11]The properties of the agents' offer curve have been previously studied to address the existence and stability of international trade equilibria (Marshall 1879 and Johnson 1958), the existence of a monetary equilibrium (see Cass et al. 1979) and endogenous business cycles (see Grandmont 1985).

feature (i.e. the monotonicity in the direction of the axes) play a crucial role for uniqueness. This sharply contrasts with the conditions on *aggregate* excess of demand guaranteeing monotonicity. In fact, these conditions on individual offer curves are, in general, weaker than those usually found in the literature (e.g., gross substitutability of the excess demand function), as monotonicity of the individual offer curves does not necessarily imply monotonicity of the aggregate excess of demand. For instance, in a CES economy it suffices to assume that only one side of the market satisfies the gross substitutability property to ensure uniqueness (as illustrated in Figure 1). Finally, we also provide explicit and testable conditions on the agents' preferences and endowments ensuring that these sufficient conditions for uniqueness are satisfied (Props.5.5 and 5.12). These conditions are illustrated within well-known utility representations of preferences commonly used in economic theory.

[Figure 1 about here.]

The paper is organized as follows. Section 2 describes the model and the individual agent's problem. Section 3 characterizes monotonicity and convexity properties of offer curves. Section 4 focuses on the properties of three particular classes of offer curves usually displayed in the literature. Section 5 provides testable conditions to guarantee that a competitive equilibrium is unique for different shapes of offer curves at each side of the market; in addition, it states a testable sufficient condition for multiplicity of equilibria. Section 6 illustrates these conditions in well-known utility representations of preferences. Section 7 concludes.

2 Assumptions, definitions and notation

Let us consider a two-commodity two-agent decentralized exchange economy, $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$. Let $L = 2$ denote the total number of commodities. Let p_l denote the price of the commodity l, with $l = 1, 2$. The price vector is $\mathbf{p} = (p_1, p_2)$, and $R = p_1/p_2$ denotes the relative price. Let $\mathcal{P} \equiv \mathcal{R}_{++}$ denote the set of strictly positive relative prices. Let $I = 2$ be the number of (types of homogeneous) agents, denoted to by A and B. All agents share the same consumption set $\mathcal{X} = \mathcal{R}^2_+$. Let $\boldsymbol{\omega}^i = (\omega^i_1, \omega^i_2) \in \mathcal{R}^2_+$ denote the vector of endowments of agent i, and $\boldsymbol{\omega} = (\boldsymbol{\omega}_1, \boldsymbol{\omega}_2) \in \mathcal{R}^2_{++}$ represents the aggregate endowment. Preferences of agent i are represented by a continuous utility function $U^i : \mathcal{X} \to \mathcal{R}$ that is smoothly monotone, smoothly strictly quasi-concave and twice continuously differentiable. A continuous and monotone preferences are *homothetic* if and only if it admits a utility function representation that is homogeneous of degree one. The agent i's *indifference set* containing allocation $\mathbf{x}$ is the set of all consumption plans that are indifferent to $\mathbf{x}$, i.e. $\{\mathbf{y} \in \mathcal{X} : U^i(\mathbf{y}) = U^i(\mathbf{x})\}$. We will find it useful to consider that, without any loss of generality, agent A is most interested in purchasing good $l = 2$, and vice versa for agent B, by restricting the agents' marginal rate of substitution at the autarkic allocation[12]

Assumption 1. $MRS^A(\boldsymbol{\omega}^A) = U^A_1(\boldsymbol{\omega}^A)/U^A_2(\boldsymbol{\omega}^A) \leq U^B_1(\boldsymbol{\omega}^B)/U^B_2(\boldsymbol{\omega}^B) = MRS^B(\boldsymbol{\omega}^B)$.

2.1 The individual agent's problem.

For each agent i (with $i = A$, B) the Walrasian demand function $d^i : \mathcal{P} \to \mathcal{X}$ is the unique solution that maximizes $U^i(\mathbf{x}^i)$ subject to the budget constraint $Rx^i_1 + x^i_2 = R\omega^i_1 + \omega^i_2$. Given the

[12]This assumption has given rise to several interpretations in the literature. In international trade literature –with commodities $l = 1$ and 2 representing the *domestic* and the *foreign good*, respectively–, this assumption means that each country is prone to be involved in international trade by buying the foreign good in exchange for the domestic good. In financial economics literature –with commodities $l = 1$ and 2 representing the *present* and the *future (contingent) good*, respectively–, this assumption means that each agent is prone to transfer wealth to the future.

assumptions, the map $d^i(R;\boldsymbol{\omega}^i)$ is a smooth bijection. We introduce some useful notation. Let $R_{\boldsymbol{\omega}^i}$ denote the relative price satisfying $\boldsymbol{\omega}^i = d^i(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)$, and let $\Psi^i_l : \mathcal{P} \to \mathcal{R}$ be the function defined for any $\boldsymbol{\omega}^i$ as

$$\Psi^i_l(R;\boldsymbol{\omega}^i) = d^i_l(R;\boldsymbol{\omega}^i)\left[-\frac{U^i_{ll}(d^i(R;\boldsymbol{\omega}^i))}{U^i_l(d^i(R;\boldsymbol{\omega}^i))} + \frac{U^i_{lk}(d^i(R;\boldsymbol{\omega}^i))}{U^i_k(d^i(R;\boldsymbol{\omega}^i))}\right] \text{ with } l,\, k = 1,\, 2 \text{ and } k \neq l.$$

The coefficient Ψ^i_l can be regarded as a multivariate relative risk aversion evaluated at the optimal allocation,[13] a characterization of the curvature of the utility function being invariant under monotone transformations of U^i. The next result, proved in the Appendix, features the slope of the demand curves for commodities $l = 1, 2$.

Lemma 2.1. The slope of the demand curves at an allocation $d^i(R;\boldsymbol{\omega}^i)$. *For any $R \in \mathcal{P}$,*

$$d^{i\prime}_1(R;\boldsymbol{\omega}^i) = -U_2(d^i)\,\frac{1 + [\omega^i_2 - d^i_2(R;\boldsymbol{\omega}^i)]\frac{\Psi^i_2(R;\boldsymbol{\omega}^i)}{d^i_2(R;\boldsymbol{\omega}^i)}}{det \mid HU^i(d^i) \mid /[U_2(d^i)]^2}\text{; and, }\; d^{i\prime}_2(R;\boldsymbol{\omega}^i) = U_1(d^i)\,\frac{1 + [\omega^i_1 - d^i_1(R;\boldsymbol{\omega}^i)]\frac{\Psi^i_1(R;\boldsymbol{\omega}^i)}{d^i_1(R;\boldsymbol{\omega}^i)}}{det \mid HU^i(d^i) \mid /[U_2(d^i)]^2},$$

with $d^i \equiv d^i(R;\boldsymbol{\omega}^i)$, and $HU^i(d^i)$ being the bordered Hessian evaluated at the optimal allocation.

We conclude this section with some terminological remarks. Recall that the terminology commonly used to distinguish types of goods in exogenous-income settings might not be appropriate in general equilibrium analysis because changes in the price of any commodity simultaneously modify the agent's income. Yet, we will find it useful to characterize commodities by studying how changes in the endogenous income affect changes in the demanded quantity of a commodity. Specifically, we will refer to a commodity l as a **normal good** [resp., an **inferior good**] at some individual optimal allocation if an increase of the agent's income –due to an increase in the relative price, R– increases [resp., decreases] its demanded quantity. Thus, in our economy, commodity l is *normal* [resp., *inferior*] for agent i at the relative price $\hat{R}$ provided $d^{i\prime}_l(\hat{R};\boldsymbol{\omega}^i) > 0$ [resp., < 0]. We also denote commodities $l = 1$ and 2 to be **gross substitutes** [resp., **gross complements**] at an individual optimal allocation, if a change in the relative price leads to an opposite [resp., analogous] direction in the change of the demanded quantity of both commodities. Accordingly, in our economy, commodities $l = 1$ and 2 are *gross substitutes* [resp., *gross complements*] at the relative price $\hat{R}$ if $d^{i\prime}_1(\hat{R};\boldsymbol{\omega}^i)d^{i\prime}_2(\hat{R};\boldsymbol{\omega}^i) < 0$ [resp., > 0].

3 The offer curve

As the ratio of prices R varies, the budget line pivots around the endowment allocation $\boldsymbol{\omega}^i$, and the demanded consumptions trace out a curve: the offer curve, the orthogonal projection of the demand function d^i onto the consumption set $\mathcal{X}$. This curve is *simple* as it does not intersect with itself because of the strict convexity of preferences, and it is *regular* and *open* because of the monotonicity of preferences.[14] Next we formally define an offer curve.

Definition 1. *Agent i's* **offer curve** *associated with the endowment $\boldsymbol{\omega}^i$ is a simple regular open arc with the regular parametric representation of class $\mathcal{C}^2$, $C^i(R;\boldsymbol{\omega}^i) = d^i(R;\boldsymbol{\omega}^i)$ with $R \in \mathcal{P}$. If a basis $\{e_1, e_2\}$ is chosen, then the curve can be represented as $C^i(R;\boldsymbol{\omega}^i) = d^i_1(R;\boldsymbol{\omega}^i)e_1 + d^i_2(R;\boldsymbol{\omega}^i)e_2$.*

[13] If the utility function is additively separable, the coefficient Ψ^i_l is the relative risk aversion evaluated at $d^i(R;\boldsymbol{\omega}^i)$.

[14] The offer curve is *regular* because the demand function does not exhibit a local maximum; that is, no ratio of prices $R \in \mathcal{P}$ satisfies $\nabla d^i(R;\boldsymbol{\omega}^i) = \left(d^{i\prime}_1(R;\boldsymbol{\omega}^i), d^{i\prime}_2(R;\boldsymbol{\omega}^i)\right) = \mathbf{0}$ due to the monotonicity of preferences.

Along the paper, we will show the usefulness of splitting any offer curve into two complemented arc segments taking the endowment allocation as one endpoint: the arc segment $\overline{C}^i(R;\boldsymbol{\omega}^i)$, which corresponds to those allocations belonging to the offer curve such that $R \in [R^i_{\boldsymbol{\omega}^i}, +\infty) \equiv \overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$; and the arc segment $\underline{C}^i(R;\boldsymbol{\omega}^i)$, which corresponds to $R \in (0, R_{\boldsymbol{\omega}^i}] \equiv \underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$.

Next, we focus on two properties of offer curves. Initially, we characterize the slope of an offer curve and study its monotonicity properties; then we define curvature and present the convexity properties of an offer curve.

3.1 Offer curve: slope

The slope of an offer curve at any (optimal) allocation coincides with the slope of the tangent line at such an allocation –i.e., the slope of the line $[(x_1,x_2)-d^i(R;\boldsymbol{\omega}^i)][\nabla d^i(R;\boldsymbol{\omega}^i)]' = 0$ at a given R (see Lipschultz 1969, p.62). This slope is a measure of the *absolute elasticity of substitution* between commodities 2 and 1, as it shows how the ratio of the changes in demand is modified as the relative price increases. The proof is straightforward by dividing the slopes of the demand curves (Lemma 2.1).

Lemma 3.1. The absolute elasticity of substitution between commodities 2 and 1. *The* ***slope*** *of the tangent line to the offer curve $C^i(R;\boldsymbol{\omega}^i)$ at (optimal) allocation $d^i(R;\boldsymbol{\omega}^i)$ is*

$$\frac{d_2^{i\prime}(R;\boldsymbol{\omega}^i)}{d_1^{i\prime}(R;\boldsymbol{\omega}^i)} = -R\frac{1+[\omega_1^i - d_1^i(R;\boldsymbol{\omega}^i)]\frac{\Psi_1^i(R;\boldsymbol{\omega}^i)}{d_1^i(R;\boldsymbol{\omega}^i)}}{1+[\omega_2^i - d_2^i(R;\boldsymbol{\omega}^i)]\frac{\Psi_2^i(R;\boldsymbol{\omega}^i)}{d_2^i(R;\boldsymbol{\omega}^i)}}. \tag{1}$$

3.2 Offer curve: monotonicity

The concept of monotonicity is a property usually associated to mappings, although it also characterizes curves. Next, we present a definition of monotonicity that considers a curve monotone if its projection on a line does not backtrack.[15]

Definition 2. *An offer curve $C^i(R;\boldsymbol{\omega}^i)$ is* **monotone in direction** v *if the intersection of the curve with any line normal to v is either an empty or a connected set.*

It will be of interest to characterize the monotonicity properties of the offer curve at particular directions, specifically at the axis-directions (i.e., e_1- and e_2-direction). To this aim, it will be useful to identify critical allocations.

Definition 3.
(i) (Emeliyanenko et al. 2009, Sec.2) An allocation $d^i(R;\boldsymbol{\omega}^i)$ is called e_1-**critical** *if $d_1^{i\prime}(R;\boldsymbol{\omega}^i) = 0$ [resp.,* e_2-**critical** *if $d_2^{i\prime}(R;\boldsymbol{\omega}^i) = 0$].*[16]
(ii) An allocation $d^i(R;\boldsymbol{\omega}^i)$ is called **non-degenerate** e_1-**critical** *if it is e_1-critical and $d_1^{i\prime\prime}(R;\boldsymbol{\omega}^i) \neq 0$ [resp.,* **non-degenerate** e_2-**critical** *if it is e_2-critical and $d_2^{i\prime\prime}(R;\boldsymbol{\omega}^i) \neq 0$].*

Observe that the slope of the tangent line to the offer curve (1) is infinite at e_1-critical [resp., zero at e_2-critical] allocations. We can now define monotonicity in the direction of the axes.[17]

[15] See Díaz-Bañez et al. (2000, Sec.1), Agarwal et al. (2002, Sec.1.1), or Fox et al. (2010, Sec.1).
[16] In the literature, e_1-*critical* and e_2-*critical* are usually termed as x-*critical* and y-*critical*.
[17] The literature has restricted the notion of *monotonicity* to exclude critical, rather than non-degenerate critical, points. For instance, Emeliyanenko et al. (2009, Sec.2) presents a stronger definition: "an e_l-monotone curve arc is a curve arc that has no e_l-critical points in the interior;" or Grandmont (1985), who states that "The curve is then 'monotone', i.e. it has no critical point." (p.1002)

Definition 4. *An offer curve $C^i(R;\boldsymbol{\omega}^i)$ is* e_l-**monotone** *(i.e., it is monotone in the direction of the basis vector e_l), if it has no non-degenerate e_l-critical allocation, with $l = 1$, 2.*
An offer curve $C^i(R;\boldsymbol{\omega}^i)$ is **locally** e_l-**monotone** *around a particular allocation $d^i(\breve{R};\boldsymbol{\omega}^i)$, if it has no non-degenerate e_l-critical allocation at some neighborhood.*
If an offer curve $C^i(R;\boldsymbol{\omega}^i)$ is e_l-**monotone** *for both $l = 1$ and 2, then the curve is* **monotone**.

The notion of the monotonicity of a curve is also related with the notion of the slope of the curve: a curve is monotonic at a connected subset of (optimal) allocations if and only if it presents no sign change in the derivative value along any part of the curve in the interval (see Wolberg et al. 2002, Sec.4). Thus, we can characterize offer curves that satisfy the *gross substitute* property as those that are downward sloping and, thus, monotone.

Definition 5. The gross substitute property. *(Mas-Colell et al. 1995, p.541.)*
(i) An offer curve satisfies the **gross substitute** *property at some allocation $d^i(R;\boldsymbol{\omega}^i)$ if an increase in the (relative) price of one commodity decreases the demand for that commodity and increases the demand for the other one; i.e., the absolute elastiticy of substitution –the slope of the offer curve– at such an allocation is negative, $d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i) < 0$.*
(ii) An offer curve satisfies the **gross substitute** *property if the offer curve is* ***monotone****; i.e., the slope of the offer curve is always negative, $d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i) < 0$ for all $R \in \mathcal{P}$.*

Next, we present two results. The first result, proved in the Appendix, characterizes the monotonicity properties of offer curves.

Theorem 3.2. *Let $C^i(R;\boldsymbol{\omega}^i)$ with $R \in \mathcal{P}$ be an offer curve.*

(*i*) *The offer curve $C^i(R;\boldsymbol{\omega}^i)$ is* ***locally monotone*** *around the allocation $d^i(R_{\boldsymbol{\omega}^i}, \boldsymbol{\omega}^i) = \boldsymbol{\omega}^i$.*

(*ii*) *Suppose that the utility function (a) is homothetic, or (b) satisfies $\Psi_1^i(R;\boldsymbol{\omega}^i)\Psi_2^i(R;\boldsymbol{\omega}^i) \geq 0$ for all $R \in \mathcal{P}$. Then the arc segments $\underline{C}^i(R;\boldsymbol{\omega}^i)$ and $\overline{C}^i(R;\boldsymbol{\omega}^i)$ are* e_1-**monotone** *and* e_2-**monotone**, *respectively.*

(*iii*) *If $\Psi_2^i(R;\boldsymbol{\omega}^i) \in [0, 1]$ for any $R \in \mathcal{P}$, then the offer curve $C^i(R;\boldsymbol{\omega}^i)$ is* e_1-**monotone**. *Analogously, if $\Psi_1^i(R;\boldsymbol{\omega}^i) \in [0, 1]$ for any $R \in \mathcal{P}$, then the offer curve $C^i(R;\boldsymbol{\omega}^i)$ is* e_2-**monotone**.

(*iv*) **Monotonicity and the negative slope of the offer curve.** *Suppose that*

$$\left[1 + \Psi_1^i(R;\boldsymbol{\omega}^i)\frac{\omega_1^i - d_1^i(R;\boldsymbol{\omega}^i)}{d_1^i(R;\boldsymbol{\omega}^i)}\right]\left[1 + \Psi_2^i(R;\boldsymbol{\omega}^i)\frac{\omega_2^i - d_2^i(R;\boldsymbol{\omega}^i)}{d_2^i(R;\boldsymbol{\omega}^i)}\right] > 0$$

is satisfied for any $R \in \mathcal{P}$ –e.g., as in the case that $\Psi_1^i(R;\boldsymbol{\omega}^i)$, $\Psi_2^i(R;\boldsymbol{\omega}^i) \in (0, 1]$ for any $R \in \mathcal{P}$. Then the offer curve is ***monotone*** *and, thus, it satisfies the* **gross substitute property**.

Part (*i*) states that the offer curve is always locally monotone around the autarkic allocation. This means that commodities $l = 1$ and 2 are *gross substitutes* at some neighborhood of the endowment allocation and, specifically, commodity $l = 1$ is an *inferior good* while commodity $l = 2$ is a *normal good*, i.e. there exists an $\varepsilon > 0$ such that $d_1^{i\prime}(R;\boldsymbol{\omega}^i) < 0$ and $d_2^{i\prime}(R;\boldsymbol{\omega}^i) > 0$ for any $R \in \mathcal{B}(R_{\boldsymbol{\omega}^i}, \varepsilon)$. Part (*ii*) shows that, although offer curves need not be monotone, any arc segment of an offer curve with endpoint at the autarkic allocation is always monotone in some direction by restricting in a certain way the properties of the utility function.[18] Part (*iii*), a

[18]For example, homotheticity, or that the utility function U^i has a monotone transformation V satisfying that the cross-derivative is positive, $V_{12}(d^i(R;\boldsymbol{\omega}^i)) > 0$ for all R (see Proposition A.1).

generalization of Grandmont (1985, Lem.1.3(i)), shows that the result (ii) can be extended to the overall offer curve by restricting the degree of concavity of the utility function.

Finally, part (iv) generalizes the sufficient condition of Hens et al. (1995, Sec.5) guaranteeing that an individual Walrasian demand satisfies the gross substitution property in the particular case of preferences represented by an additively separable utility function. The condition in (iv) also guarantees the negativity of the slope of an offer curve (1) –and then its monotonicity–, and can also be displayed as

$$-\frac{[\boldsymbol{\omega}^i - d^i(R;\boldsymbol{\omega}^i)]HU^i(d^i(R;\boldsymbol{\omega}^i))[\boldsymbol{\omega}^i - d^i(R;\boldsymbol{\omega}^i)]'}{[\omega_1^i - d_1^i(R;\boldsymbol{\omega}^i)]U_1(R;\boldsymbol{\omega}^i)\left[1+\Psi_1^i(R;\boldsymbol{\omega}^i)\frac{\omega_1^i - d_1^i(R;\boldsymbol{\omega}^i)}{d_1^i(R;\boldsymbol{\omega}^i)}\right]} < 1$$

for any $R \in \mathcal{P}$. This condition differs from the necessary and sufficient condition that guarantee the law of demand found in Milleron (1974), Mitjuschin and Polterovich (1978) and Quah (2000), for the following two reasons: we consider that income is price dependent, instead of price independent; and, their condition involves a curvature condition *for all* the allocations in the interior of the consumption set, while ours involves a monotone condition *at the optimal condition*, i.e. the allocations belonging to the offer curve defined for a given endowment vector $\boldsymbol{\omega}^i$.

Our second result characterizes two relevant critical points. Note that, as a consequence of Lemma 3.1 Theorem 3.2.(iv), if a non-degenerate critical allocation exists, then an offer curve cannot be monotone. Yet, the offer curve is locally monotone at the autarkic allocation (Theorem 3.2.(i)), so the offer curve stops being monotone if there exists either a non-degenerate e_1-critical allocation at some $\widehat{R} > R_{\boldsymbol{\omega}^i}$, or a non-degenerate e_2-critical allocation at some $\widetilde{R} < R_{\boldsymbol{\omega}^i}$. The proof is in the appendix.

Proposition 3.3. ***Characterization of non-degenerate*** e_l-***critical allocations.*** *Consider any arc segment identified by a subarc of the offer curve* $C^i(R;\boldsymbol{\omega}^i)$ *containing the autarkic allocation; that is, a subarc of* $C^i(R;\boldsymbol{\omega}^i)$ *with* $R \in [\widetilde{R}, R_{\boldsymbol{\omega}^i}] \bigcup [R_{\boldsymbol{\omega}^i}, \widehat{R}] \subseteq \mathcal{P}$. *Suppose that such an arc segment is monotone –i.e. no interior allocation of the arc segment is non-degenerate* e_1- *or non-degenerate* e_2-*critical.*
(i) If there exists one non-degenerate e_1-*critical allocation at a subarc* $C^i(R;\boldsymbol{\omega}^i)$ *with* $R \in [\widetilde{R}, \widehat{R}]$, *then it can only be the optimal allocation defined by the relative price* $\widehat{R} > R_{\boldsymbol{\omega}^i}$, *i.e.* $d_1^{i\prime}(\widehat{R};\boldsymbol{\omega}^i) = 0$ *and* $d_1^{i\prime\prime}(\widehat{R};\boldsymbol{\omega}^i) \neq 0$*; and,*
(ii)If there exists a non-degenerate e_2-*critical allocation at the subarc* $C^i(R;\boldsymbol{\omega}^i)$ *with* $R \in [\widetilde{R}, \widehat{R}]$, *then it can only be the optimal allocation defined by the relative price* $\widetilde{R} < R_{\boldsymbol{\omega}^i}$, *i.e.* $d_2^{i\prime}(\widetilde{R};\boldsymbol{\omega}^i) = 0$ *and* $d_2^{i\prime\prime}(\widetilde{R};\boldsymbol{\omega}^i) \neq 0$.

Beyond any of these critical allocations –if they turn out to exist– the offer curve stops satisfying the gross substitute property. The income effect becomes higher than the substitution effect for some commodity and the two commodities become *gross complements* for an arc segment of the offer curve. We formally define this property. Unlike the usual definition (e.g., Mas-Colell et al. 1995, p.70), we are defining that commodities $l = 1$ and 2 are *gross complements* at some allocation if both demands move in the *same* direction as the (relative) price is modified.

Definition 6. The gross complement property. *An offer curve satisfies the* **gross complement** *property at some allocation* $d^i(R;\boldsymbol{\omega}^i)$ *if an increase in the relative price of one commodity simultaneously increases or decreases the demand for both commodities; i.e., the elastiticy of substitution –the slope of the offer curve– at such an allocation is positive,* $d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i) > 0$.

As a final comment, it is worth noting that non-degenerate critical allocations need not exist, so an offer curve can be monotone. This could sound striking since the monotonicity of preferences implies that it is optimal to consume higher amounts of a commodity as it becomes cheaper (i.e., as the relative prices goes to zero or to infinite). Then, one might expect that, in the limit, the optimal individual behavior consists in demanding infinite units of one of the commodities at zero price without giving up any unit of the other commodity, which requires that the offer curve must exhibit a positive slope for a sufficiently high or low price. Yet, the behavior just described might not be optimal because the demand function need not be continuous at $R = 0$ or $R = \infty$. As shown in Section 6, a well-known counterexample arises with Cobb-Douglas preferences, for which the offer curve is monotone.

3.3 Offer curve: curvature

One of the basic problems in geometry is to exactly determine the geometric features that distinguish one figure from another, a problem that can generally be solved for sufficiently smooth regular curves (see Lipschultz 1969, Chap.4). Any regular curve, such as an offer curve, is uniquely determined by two scalar quantities: the first curvature, or simply *curvature*, and the second curvature or *torsion.* Yet, the latter requires that the offer curve be a regular parametric representation of class $\mathcal{C}^3$, which involves properties of the third derivatives of the utility function with no axiomatic foundation in the standard theory of preferences. Thus, we will only focus on the (first) curvature of the offer curves and its properties.

Definition 7. *The* **curvature vector** *to the offer curve* $C^i(R;\boldsymbol{\omega}^i)$ *at allocation* $d^i(R;\boldsymbol{\omega}^i)$ *is*

$$\mathbf{k}^i(R;\boldsymbol{\omega}^i) \quad = \quad \frac{d_1^{i\prime\prime}(R;\boldsymbol{\omega}^i)d_2^{i\prime}(R;\boldsymbol{\omega}^i) - d_1^{i\prime}(R;\boldsymbol{\omega}^i)d_2^{i\prime\prime}(R;\boldsymbol{\omega}^i)}{\|d^{i\prime}(R;\boldsymbol{\omega}^i)\|^2}\left[d_2^{i\prime}(R;\boldsymbol{\omega}^i)e_1 - d_1^{i\prime}(R;\boldsymbol{\omega}^i)e_2\right].$$

Note that an allocation belonging the offer curve for which the curvature vector satisfies $\mathbf{k}^i(R;\boldsymbol{\omega}^i) = \mathbf{0}$ is called a *point of inflection.* If the offer curve is not e_1- or e_2-monotone, then one candidate to point of inflection is an optimal allocation where the offer curve is tangent to the budget line. In this case, there would exist a relative price $\breve{R}$ such that $d_2^{i\prime}(\breve{R};\boldsymbol{\omega}^i)/d_1^{i\prime}(\breve{R};\boldsymbol{\omega}^i) = -\breve{R}$. This case is ruled out by the strict convexity of preferences, as shown in the next result proved in the Appendix.

Proposition 3.4. *There exists no allocation* $d^i(R;\boldsymbol{\omega}^i)$ *with* $R \in \mathcal{P}\backslash\{R_{\boldsymbol{\omega}^i}\}$ *such that the offer curve* $C^i(R;\boldsymbol{\omega}^i)$ *is tangent to the budget line.*

3.4 Offer curve: convexity

The notion of convexity has been extended to characterize curves, attending to geometric or curvature features. The first definition, owed to Barner (1956), makes use of a geometrical feature.[19]

Definition 8. A Barner curve. *A curve in* $\mathcal{R}^n$ *is* **strongly convex** *(or* **strictly convex***) if at least one hyperplane goes through any* $n-1$ *points of the curve containing no other points of the curve.*

[19] Fabricius-Bjeree (1961, 1964) present a definition closely related to Barner's: an open curve is *linearly monotone* if the common points for the curve and an arbitrary line in the plane form the same sequence on the curve and the line. This notion is similar to the concept of *coplanar convexity* proposed by Labenski et al. (1996). Other definitions on geometric grounds have been proposed: a curve is convex provided it never crosses a straight line more than twice (Schoenberg 1954); a curve in $\mathcal{R}^n$ is said to be *convex* in $\mathcal{R}^n$ "if any hyperplane divides the curve into no more than n parts," or "if any hyperplane does not intersect the curve more than n times, taking multiplicities into account." (e.g., Anisov 1998).

This notion will be shown to be useful in characterizing offer curves, although it is not a space-oriented concept of convexity. The second definition is based on the notion of curvature.

Definition 9.
(i) (Liu et al. 1997, Th.3.1). Let be an offer curve $C^i(R;\boldsymbol{\omega}^i)$ satisfying $\mathcal{C}^2-$continuous. The offer curve $C^i(R;\boldsymbol{\omega}^i)$ is ***locally convex*** *around an allocation $d^i(\breve{R};\boldsymbol{\omega}^i)$ if, and only if,*

$$d_1^{i\prime}(\breve{R};\boldsymbol{\omega}^i)d_2^{i\prime\prime}(\breve{R};\boldsymbol{\omega}^i) - d_1^{i\prime\prime}(\breve{R};\boldsymbol{\omega}^i)d_2^{i\prime}(\breve{R};\boldsymbol{\omega}^i) \leq 0; \tag{2}$$

(ii) The offer curve $C^i(R;\boldsymbol{\omega}^i)$ is a ***globally convex*** *curve if (2) satisfies for all $R \in \mathcal{P}$.*

The following result, proved in the Appendix, refers to local convexity.

Proposition 3.5.
(i) Any offer curve $C^i(R;\boldsymbol{\omega}^i)$ is ***locally convex*** *around the allocation $d^i(R_{\boldsymbol{\omega}^i},\boldsymbol{\omega}^i) = \boldsymbol{\omega}^i$; and,*
(ii) Consider the monotone arc segment defined in Proposition 3.3. Then, the offer curve $C^i(R;\boldsymbol{\omega}^i)$ is ***locally convex*** *around non-degenerate e_1- and non-degenerate e_2-critical allocations, provided they exist.*

Note that local convexity at an allocation belonging to an offer curve is equivalent to the fact that there exists a sufficiently small neighborhood of such allocation that always lies at the right side of the tangent line (see Liu et al. 1997, Def.2.4). Yet, this tangent line is a local supporting line of the oriented offer curve at the allocation (see Carnicer et al. 1998, Def.2.5), and it is easy to see that the entire offer curve need not lie at the right side.

Finally, we present two results concerning global convexity. The first result links the convexity properties of the demand function $d^i(R;\boldsymbol{\omega}^i)$ with those of the offer curve. Since the offer curve is a projection of the demand function, convexity becomes an invariant property.[20]

Proposition 3.6. *(Anisov 1998, Lemma 2). If the demand function d^i is convex, then the offer curve $C^i(R;\boldsymbol{\omega}^i)$ is globally convex.*

Our second result, proved in the Appendix, states that the set of globally convex offer curves is a subset of the set of monotone offer curves.

Proposition 3.7. *If the offer curve $C^i(R;\boldsymbol{\omega}^i)$ is* ***globally convex****, then it is* ***monotone****.*

The reverse might not be true: if there exists a non-degenerate critical allocation, an offer curve cannot be globally convex. In light of these propositions, global convexity is a very restrictive property when applied to offer curves, thus reducing its applicability.

4 Three classes of offer curves

In this section, we explore the properties of three profiles of offer curves proposed by the literature (see Figure 2).[21] Two well-known types of offer curves, *gross substitute* and *normal* offer curves, are characterized from the properties of local and global monotonicity. In addition, we depict *non-normal* offer curves.

[Figure 2 about here.]

[20]Carnicer et al. (1996) also presented geometric conditions for monotonocity and convexity of a function as invariant properties under projections.

[21]We will not consider a fourth shape proposed by Johnson (1958, Fig.6), because the agent's preferences fail to be transitive.

4.1 Shape 1: Gross substitute offer curves

Initially, we characterize *gross substitute offer curves* from previous results. A *gross substitute offer curve* $C^i(R;\boldsymbol{\omega}^i)$ satisfies the *gross substitute* property for all $R \in \mathcal{P}$ (see Definition 5) and displays the shape pictured in Figure 2(a) in a two-commodity economy. It always displays a slope with a negative sign (Definition 5.(ii)) and presents no non-degenerate critical allocations (Proposition 3.3). A condition to identify *gross substitute* offer curves are found in Theorem 3.2.(iv). Such a condition is indeed satisfied for $\Psi_1^i(R;\boldsymbol{\omega}^i)$, $\Psi_2^i(R;\boldsymbol{\omega}^i) \in (0,1]$ for any $R \in \mathcal{P}$, as in the constant –and less than or equal to one–, relative risk aversion utility functions.[22] Interestingly, the set of globally convex offer curves belongs to the class of offer curves satisfying the gross substitute property (Proposition 3.7), although the converse is not true. We will present further characterizations in Section 4.3.

4.2 Shape 2: Normal offer curves

In this section we characterize a particular class of offer curves, those denoted as *normal. Normal offer curves* necessarily require the existence of non-degenerate critical allocations, so that commodities stop being gross substitutes and become gross complements (or viceversa) beyond critical allocations.

The simplest illustration is the case depicted in Figure 2(b), the case of a *normal offer curve* with one non-degenerate e_1-critical allocations for the relative price $\widehat{R} > R_{\boldsymbol{\omega}^i}$. Beyond this threshold, the slope of the offer curve changes its sign: the slope $d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i)$ is negative for $R < \widehat{R}$ –so commodities $l = 1$ and 2 are *gross substitutes*–, and the slope becomes positive for relative prices above $\widehat{R}$ –i.e., commodities are *gross complements.* We begin by defining and exploring *normal offer curves* displaying only one non-degenerate critical allocation (i.e., one of those identified in Proposition 3.3).

Definition 10.
*(i) An offer curve $C^i(R;\boldsymbol{\omega}^i)$ is denoted to as $l = 1-$**normal** if there exists a relative price $\widetilde{R} < R_{\boldsymbol{\omega}^i}$ that corresponds to a non-degenerate e_2-critical allocation $d^i(\widetilde{R},\boldsymbol{\omega}^i)$, such that it is satisfied $[d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i)][R-\widetilde{R}] \leq 0$ for any $R \in \mathcal{P}$;*
*(ii) An offer curve $C^i(R;\boldsymbol{\omega}^i)$ is denoted to as $l = 2-$**normal** if there exists a relative price $\widehat{R} > R_{\boldsymbol{\omega}^i}$ that corresponds to a non-degenerate e_1-critical allocation $d^i(\widehat{R},\boldsymbol{\omega}^i)$, such that it is satisfied $[d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i)][\widehat{R}-R] \leq 0$ for any $R \in \mathcal{P}$.*

Thus, an $l = 1-$**normal** [resp., $l = 2-$**normal**] **offer curve** $C^i(R;\boldsymbol{\omega}^i)$ satisfies the *gross substitute* property at any allocation for every $R > \widetilde{R}$ [resp., $R < \widehat{R}$], and the *gross complement* property at any allocation for every $R < \widetilde{R}$ [resp., $R > \widehat{R}$]. Observe that $l-$*normal offer curves* satisfy that commodity l is a normal good at every (optimal) allocation of the offer curve,[23] due to an unequivocal sign of the income effect. The following result, proved in the Appendix, characterizes $l-$*normal offer curves.*

Proposition 4.1. *Consider an $l = 1-$**normal** [resp., $l = 2-$**normal**] **offer curve** $C^i(R;\boldsymbol{\omega}^i)$ with the non-degenerate e_2-critical allocation $d^i(\widetilde{R};\boldsymbol{\omega}^i)$ at $\widetilde{R} \in \underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$ [resp., the non-degenerate*

[22]For instance, the Cobb-Douglas utility function with $\Psi_1^i(R;\boldsymbol{\omega}^i) = \Psi_2^i(R;\boldsymbol{\omega}^i) = 1$ for any $R \in \mathcal{P}$, or the CES utility function for particular welfare parameters (namely, $\Psi_1^i(R;\boldsymbol{\omega}^i) = \Psi_2^i(R;\boldsymbol{\omega}^i) = 1-\rho$ with $\rho \geq 0$ for any $R \in \mathcal{P}$ under the specification in Section 6, Ex.2).

[23]Although it could not be reported with a reference from the literature, the term *normal* offer curve probably stems from this property if the offer curve present only one non-degenerate critical allocation.

e$_1-$*critical allocation* $d^i(\widehat{R};\boldsymbol{\omega}^i)$ *with* $\widehat{R}\in\overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$]. *Then,*
(i) the offer curve is e$_1-$*monotone [resp.,* e$_2-$*monotone];*
(ii) the two complement arc segments of the offer curve $C^i(R;\boldsymbol{\omega}^i)$ *satisfying* $R>\widetilde{R}$ *and* $R<\widetilde{R}$ *are both* e$_2-$*monotone [resp.,* $R<\widehat{R}$ *and* $R>\widehat{R}$ *are both* e$_1-$*monotone] –i.e. in the former the commodities are gross substitutes, while in the latter they are gross complements–;*
(iii) the offer curve is **locally convex around the non-degenerate critical allocation** $d^i(\widetilde{R};\boldsymbol{\omega}^i)$ *[resp.,* $d^i(\widehat{R};\boldsymbol{\omega}^i)$*]; and,*
(iv) one allocation $d^i(\breve{R};\boldsymbol{\omega}^i)$ *with* $\breve{R}<\widehat{R}$ *[resp.,* $\breve{R}>\widehat{R}$*] must be a* **point of inflection** *of the offer curve.*

Observe that we have focused on exploring the properties of those offer curves exhibiting only one non-degenerate critical allocation. Yet, it is not difficult to trace normal offer curves with more than one. The conditions for the existence of only one critical allocation, however, require constraining the second curvature of the utility function –with its third derivative involved–, which falls out of the realm of modern microeconomic theory as it involves cardinal welfare profiles. Indeed, Grandmont's (1985) Lemma 3.1(ii) –which guarantees a unique critical allocation by restricting the monotonicity of the Arrow-Pratt relative degrees of risk aversion– is not immune to this critique.[24] In the Appendix, we prove the following extension of Grandmont's result to our setting.

Lemma 4.2.
(i) If $\Psi_2^i(R;\boldsymbol{\omega}^i)$ *is non-decreasing for all* $R\in\overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$ *and* $\Psi_2^i(R;\boldsymbol{\omega}^i)>1$ *for some* $R\in\overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$*, then there exists a* **unique** *non-degenerate* e$_1-$*critical allocation* $d^i(\widehat{R};\boldsymbol{\omega}^i)$ *with* $\widehat{R}\in\overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$*; analogously,*
(ii) if $\Psi_1^i(R;\boldsymbol{\omega}^i)$ *is non-increasing for all* $R\in\underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$ *and* $\Psi_1^i(R;\boldsymbol{\omega}^i)>1$ *for some* $R\in\underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$*, then there exists a* **unique** *non-degenerate* e$_2-$*critical allocation* $d^i(\widetilde{R};\boldsymbol{\omega}^i)$ *with* $\widetilde{R}\in\underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$*.*

To conclude the section, we explore more general normal offer curves. For instance, offer curves displaying two non-degenerate critical allocations, those characterized in Proposition 3.3 (see Figure 2(c)). In this case, Proposition 4.1 holds for each complement arc segment of the offer curve, $\overline{C}^i(R;\boldsymbol{\omega}^i)$ and $\underline{C}^i(R;\boldsymbol{\omega}^i)$. We can also present a general characterization of *normal offer curves* with more than one non-degenerate e_l-critical allocation.

Definition 11. *(Mas-Colell et al. 1995, p.541.) An offer curve is denoted as* **normal** *if an increase in the (relative) price of one commodity leads to an increase in the demand for that commodity only if the demands of the two commodities both increase; i.e., there exists* $\overline{J}$, $\underline{J}\geq 0$ *relative prices,* $\widetilde{R}_{\underline{J}}<...<\widetilde{R}_2<\widetilde{R}_1<R_{\boldsymbol{\omega}^i}<\widehat{R}_1<\widehat{R}_2<...<\widehat{R}_{\overline{J}}$ *with* $\overline{J}+\underline{J}>0$*, such that*

$$\frac{d_2^{i\prime}(R;\boldsymbol{\omega}^i)}{d_1^{i\prime}(R;\boldsymbol{\omega}^i)}\prod_{j=1}^{\overline{J}}(\widehat{R}_j-R)\prod_{j=1}^{\underline{J}}(R-\widetilde{R}_j)\leq 0$$

holds for any $R\in\mathcal{P}$*, with each* $\widehat{R}_j\in\overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$ *[resp.,* $\widetilde{R}_j\in\underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$*] corresponding to a non-degenerate* e$_1-$*critical allocation* $d^i(\widehat{R}_j,\boldsymbol{\omega}^i)$ *[resp., a non-degenerate* e$_2-$*critical allocation* $d^i(\widetilde{R}_j,\boldsymbol{\omega}^i)$*].*

In the following Proposition, straightforward from Definition 4, monotonicity shapes normal offer curves. In addition, the result shows that in normal offer curves every commodity l might fail to be a *normal good* for every $R\in\mathcal{P}$. Yet, commodity $l=1$ [resp., $l=2$] is unequivocally a *normal good* at the arc segment $\underline{C}^i(R;\boldsymbol{\omega}^i)$ [resp. $\overline{C}^i(R;\boldsymbol{\omega}^i)$].

[24]See, however, an argumentative interpretation of the non-decreasing absolute risk aversion as a *positive precautionary demand for saving* in Leland (1968, pp.468-470), Kimball (1990) and, more recently, Geanokoplos et al. (2018, Introduction).

Proposition 4.3. *If an offer curve $C^i(R;\boldsymbol{\omega}^i)$ is normal, then it satisfies one of the following: (i) the offer curve is e_1-monotone; (ii) it is e_2-monotone; or, (iii) $\overline{C}^i(R;\boldsymbol{\omega}^i)$ is e_2-monotone and $\underline{C}^i(R;\boldsymbol{\omega}^i)$ is e_1-monotone.*

4.3 Characterization of gross substitute and normal offer curves

Next, we present two results characterizing gross substitute and normal offer curves under particular assumptions. In the first result, both the strict convexity and homotheticity of preferences imply that any offer curve must be a gross substitute or a normal one. The proposition is proved in the Appendix, and it is a consequence of Theorem 3.2.(*ii*).

Proposition 4.4.
(i) If $\Psi_1^i(R;\boldsymbol{\omega}^i)\Psi_2^i(R;\boldsymbol{\omega}^i) \geq 0$ is satisfied for all $R \in \mathcal{P}$, then the offer curve either satisfies the gross substitute property or it is normal.
(ii) If the utility function is homothetic, then (a) the offer curve either satisfies the gross substitute property or it is normal; and, (b) the class of offer curves belongs to the set of Barner curves, i.e. they are strictly convex.

The second result considers the case of additively separable utility functions, i.e., those represented by constant-relative-risk-aversion, quadratic or quasilinear continuous utility functions, or the Bernoulli utility function commonly used to represent preferences in financial economics (see, e.g., Hens et al. 1995 or Geanakoplos et al. 2018). The first part of the result is a consequence of Proposition 4.4.(*i*), since $\Psi_1^i(R;\boldsymbol{\omega}^i)$ and $\Psi_2^i(R;\boldsymbol{\omega}^i)$ are always positive for any $R \in \mathcal{P}$. The second part of the corollary is a consequence of Theorem 3.2.(*iii*).

Lemma 4.5. *Suppose an additively separable preferences represented by the continuous utility function $U^i(\mathbf{x}^i) = \sum_{l=1}^{L} \lambda_l u(x_l^i)$ with $\lambda_l > 0$ and $l = 1,..., L$. Then, the offer curve either satisfies the gross substitute property or it is normal. If, in addition, $\Psi_l^i(R;\boldsymbol{\omega}^i) \in [0,1]$ is satisfied for all $R \in \mathcal{P}$ and any $l = 1,...L$, then the offer curve satisfies the* ***gross substitute*** *property.*

4.4 Shape 3: Non-normal offer curves

Proposition 4.4 restricts the possibility for an offer curve to take the shape drawn at Figure 2(d) to the case that the preference ordering has a continuous representation U that (i) is not homothetic, (ii) is not additively separable, or (iii) is strictly quasiconcave satisfying $U_{12}(\mathbf{x}) > 0$ for $R \in \mathcal{P}$.[25] This shape requires that either the arc segment $\overline{C}(R;\boldsymbol{\omega}^i)$ not be e_2-monotone or that the arc segment $\underline{C}(R;\boldsymbol{\omega}^i)$ not be e_1-monotone –i.e., Proposition 4.3 is not satisfied–, so there must exist two or more critical allocations at one (or both) of these complement arc segments, as stated in the following condition.

Lemma 4.6. *A necessary condition for an offer curve to be non-normal is (a) there exists a non-degenerate e_2-critical allocation satisfying $d_2^{i\prime}(R;\boldsymbol{\omega}^i) = 0$ for some $R \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$; or, (b) there exists a non-degenerate e_1-critical allocation satisfying $d_1^{i\prime}(R;\boldsymbol{\omega}^i) = 0$ for some $R \in \underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$.*

Next, we characterize a particular non-normal offer curve usually depicted in the literature as in Figure 2(d).

[25] That is, there is no continuous representation displaying *non-normal* offer curves like the one depicted in Figure 2(d) such that $U_{12}(\mathbf{x}) < 0$, as shown in Proposition A.1 in the Appendix.

Proposition 4.7. *Consider the arc segment* $\overline{C}^i(R;\boldsymbol{\omega}^i)$ *with two non-degenerate critical allocations* $d^i(\widehat{R}_1;\boldsymbol{\omega}^i)$ *and* $d^i(\widehat{R}_2;\boldsymbol{\omega}^i)$*, such that* $d_1^{i\prime}(\widehat{R}_1;\boldsymbol{\omega}^i)=0$ *and* $d_2^{i\prime}(\widehat{R}_2;\boldsymbol{\omega}^i)=0$ *[resp.,* $\underline{C}^i(R;\boldsymbol{\omega}^i)$ *with* $d^i(\widetilde{R}_1;\boldsymbol{\omega}^i)$ *and* $d^i(\widetilde{R}_2;\boldsymbol{\omega}^i)$*, such that* $d_2^{i\prime}(\widetilde{R}_1;\boldsymbol{\omega}^i)=0$ *and* $d_1^{i\prime}(\widetilde{R}_2;\boldsymbol{\omega}^i)=0$*]. Then,*
(i) the arc segment of the offer curve $C^i(R;\boldsymbol{\omega}^i)$ *with* $R\in[R_{\boldsymbol{\omega}^i},\widehat{R}_2]$ *is* e_2-*monotone [resp.,* $R\in[\widetilde{R}_2,R_{\boldsymbol{\omega}^i}]$ *is* e_1-*monotone];*
(ii) the arc segment of the offer curve $C^i(R;\boldsymbol{\omega}^i)$ *with* $R>\widehat{R}_1$ *is* e_1-*monotone [resp.,* $R\le\widetilde{R}_1$ *is* e_2-*monotone];*
(iii) there must exists an additional critical allocation $d^i(\widehat{R}_3;\boldsymbol{\omega}^i)$ *with* $\widehat{R}_3>\widehat{R}_2$ *such that* $d_2^{i\prime}(\widehat{R}_3;\boldsymbol{\omega}^i)=0$ *[resp.,* $d^i(\widetilde{R}_3;\boldsymbol{\omega}^i)$ *with* $\widetilde{R}_3<\widetilde{R}_2$ *such that* $d_1^{i\prime}(\widetilde{R}_3;\boldsymbol{\omega}^i)=0$*];*
(iv) the offer curve $C^i(R;\boldsymbol{\omega}^i)$ *is locally convex around the critical allocations* $d^i(\widehat{R}_1;\boldsymbol{\omega}^i)$ *and* $d^i(\widehat{R}_2;\boldsymbol{\omega}^i)$*, but not around* $d^i(\widehat{R}_3;\boldsymbol{\omega}^i)$ *[resp.,* $d^i(\widetilde{R}_1;\boldsymbol{\omega}^i)$ *and* $d^i(\widetilde{R}_2;\boldsymbol{\omega}^i)$*, but not around* $d^i(\widetilde{R}_3;\boldsymbol{\omega}^i)$*]; and,*
(v) there must exists at least an allocation $d^i(\widehat{R};\boldsymbol{\omega}^i)$ *with* $\widehat{R}\in(\widehat{R}_2,\widehat{R}_3)$ *[*$d^i(\widetilde{R};\boldsymbol{\omega}^i)$ *with* $\widetilde{R}\in(\widetilde{R}_3,\widetilde{R}_1)$*] being a point of inflection of the offer curve.*

This result establishes a particular disposition of the non-degenerate critical allocations along the offer curve satisfying $R^i_{\boldsymbol{\omega}_i}<\widehat{R}_1<\widehat{R}_2<\widehat{R}_3$ [resp., $R^i_{\boldsymbol{\omega}_i}>\widetilde{R}_1>\widetilde{R}_2>\widetilde{R}_3$]. Note also that a *non-normal* offer curve of the type depicted in Proposition 4.7 is characterized by the type of goods that features the individual demand function. Both commodities are gross substitutes at those allocations $d^i(R;\boldsymbol{\omega}^i)$ such that $R\in(R^i_{\boldsymbol{\omega}_i},\widehat{R}_1)\bigcup(\widehat{R}_2,\widehat{R}_3)$ [resp., $R\in(\widetilde{R}_3,\widetilde{R}_2)\bigcup(\widetilde{R}_1,R^i_{\boldsymbol{\omega}_i})$] (Proposition 3.3 and Definition 5.(ii)); otherwise commodities are gross complements. In addition, as a consequence of Proposition 4.7.(ii), commodity $l=1$ is a normal good at those allocations $d^i(R;\boldsymbol{\omega}^i)$ such that $R>\widehat{R}_1$ [resp., $l=2$ is a normal good at $R<\widetilde{R}_1$], due to an unequivocal sign of the income effect. Finally, commodity $l=2$ becomes an inferior good at those allocations $d^i(R;\boldsymbol{\omega}^i)$ such that $R\in(\widehat{R}_2,\widehat{R}_3)$ [resp., $l=1$ is an inferior good at $R\in(\widetilde{R}_3,\widetilde{R}_2)$]. So the crucial feature for non-normal offer curves is that both commodities are *gross substitutes* at those allocations $d^i(R;\boldsymbol{\omega}^i)$ with $R\in(\widehat{R}_2,\widehat{R}_3)$ while commodity $l=2$ is an inferior good [resp., with $R\in(\widetilde{R}_3,\widetilde{R}_2)$ while commodity $l=1$ is an inferior good]. That is why the curve outlined in Figure 2(d) could be better depicted as $l=2-$normal-inferior-normal offer curve.

5 Uniqueness of the Competitive Equilibrium: Sufficient Conditions

In this section, we establish sufficient conditions to guarantee uniqueness of the competitive equilibrium by restricting the support of the distributions of individuals' preferences and endowments that characterize the agents' offer curve. The sufficient conditions are found considering that each side of the market is represented by a shape of offer curves analyzed in the previous section. Interestingly, our results differ from those presented in the literature constraining the properties of the *aggregate excess demand function*, i.e., $z(R;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B)=(z_1(R;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B),z_2(R;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B))=[d^A(R;\boldsymbol{\omega}^A)-\boldsymbol{\omega}^A]+[d^B(R;\boldsymbol{\omega}^B)-\boldsymbol{\omega}^B]$.

Initially, recall that any allocation –other than the endowment allocation– that simultaneously belongs to all market participants' offer curves corresponds to a competitive equilibrium. Specifically, the Edgeworth box representation for the two-agent, two-commodity case enables one to identify competitive equilibria at any intersection of the agents' offer curve $\mathcal{C}^A(R;\boldsymbol{\omega}^A)=C^A(R;\boldsymbol{\omega}^A)$ and $\mathcal{C}^B(R;\boldsymbol{\omega}^B)=\boldsymbol{\omega}-C^B(R;\boldsymbol{\omega}^B)$, considering both with respect to the first coordinate system.[26] The calligraphic symbol represents a curve considered with respect to the first

[26]In the Edgeworth box, the offer curves $C^A(R;\boldsymbol{\omega}^A)$ and $C^B(R;\boldsymbol{\omega}^B)$ are defined, respectively, with respect to the first coordinate system at the southwest vortex and to the second coordinate system at the northeast vortex.

coordinate system attached to agent A. In addition, observe that it is enough to restrict our analysis to the *trading* (*relative*) *price set*, that is, to the set of relative prices that might correspond to a competitive equilibrium. If Assumption 1 is satisfied, the trading price set becomes $\widehat{\mathcal{P}} \equiv [R^A_{\boldsymbol{\omega}^A}, R^B_{\boldsymbol{\omega}^B}] = \overline{\mathcal{P}}_{\boldsymbol{\omega}^A} \bigcap \underline{\mathcal{P}}_{\boldsymbol{\omega}^B}$, and our interest is restricted to the arc segments $\widehat{\mathcal{C}}^A(R;\boldsymbol{\omega}^A)$ and $\widehat{\mathcal{C}}^B(R;\boldsymbol{\omega}^B)$ for each $R \in \widehat{\mathcal{P}}$ –which are, respectively, connected subsets of the arc segments $\overline{\mathcal{C}}^A(R;\boldsymbol{\omega}^A)$ with $R \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^A}$ and $\underline{\mathcal{C}}^B(R;\boldsymbol{\omega}^B)$ with $R \in \underline{\mathcal{P}}_{\boldsymbol{\omega}^B}$.

5.1 Case I. Uniqueness with gross substitute and normal offer curves

Our first main result, proved in the Appendix, shows that a sufficient condition for uniqueness of a competitive equilibrium is that both offer curves be monotone in the direction of one of the axis.

Theorem 5.1. ***Uniqueness for*** e_l-***monotone offer curves at the trading price set.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$*, satisfying Assumption 1. If the arc segments of agent A's and B's offer curve,* $\widehat{\mathcal{C}}^A(R,\boldsymbol{\omega}^A)$ *and* $\widehat{\mathcal{C}}^B(R,\boldsymbol{\omega}^B)$*, respectively, are* simultaneously *either* e_1-*,* e_2-*monotone or both, then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price* $R^* \in \widehat{\mathcal{P}}$*.*

To test uniqueness, however, it could suffice to check that the agents' offer curve are simultaneously either e_1- or e_2-monotone. Specifically, Theorem 5.1 holds in the case that both offer curves $\widehat{\mathcal{C}}^A(R,\boldsymbol{\omega}^A)$ and $\widehat{\mathcal{C}}^B(R,\boldsymbol{\omega}^B)$ either satisfy the *gross substitute* property, or one of them is *normal* and the other satisfies the *gross substitute* property, as shown in the following result (see Figures 3(a)-3(b)).

[Figure 3 about here.]

Corollary 5.2. (i) ***Uniqueness for gross substitute offer curves.*** *If each agent's offer curve satisfies the gross substitute property, then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price* $R^* \in \widehat{\mathcal{P}}$*.*
(ii) ***Uniqueness for gross substitute and*** $l-$***normal offer curves.*** *If one agent's offer curve satisfies the gross substitute property, while the other's is* $l-$*normal with either* $l = 1$ *or 2, then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price* $R^* \in \widehat{\mathcal{P}}$*.*

Part (i) is a well-known result (for instance, Eisenberg 1961, Th.1 for the Cobb-Douglas case). Note that if all the agents' offer curve satisfy the *gross substitute* property, then the aggregate excess demand function also satisfies the gross substitute property (although the converse need not hold); thus, the excess demand function is negatively sloped, guaranteeing uniqueness (Mas-Colell et al. 1995, Prop.17.F.3). Observe that the aggregate excess demand is negatively sloped for all relative prices $R \in \mathcal{P}$, not just for a subset of the relative prices $R \in \widehat{\mathcal{P}}$ (Theorem 5.1). Part (ii), however, does not require the aggregate excess demand function to be negatively sloped for the whole relative price set $\mathcal{P}$ (see, e.g., Figure 1); yet, competitive equilibrium is unique.

We conclude the section by presenting a necessary condition that restricts the offer curve slopes at the unique equilibrium allocation, in the case that offer curves are monotone in *some* direction. Accordingly, we relax the requirement of *simultaneous* e_l-monotonicity for both offer curves assumed in Theorem 5.1 The proof is in the Appendix.

Proposition 5.3. ***Uniqueness for monotone offer curves at the trading price set: a necessary condition.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$*,* *satisfying Assumption 1. If the arc segments of agent A's and B's offer curve,* $\widehat{\mathcal{C}}^A(R, \boldsymbol{\omega}^A)$ *and* $\widehat{\mathcal{C}}^B(R, \boldsymbol{\omega}^B)$*, respectively, are* e_1-*,* e_2-*monotone or both, then a unique competitive equilibrium allocation, with an equilibrium relative price* $R^* \in \widehat{\mathcal{P}}$*, must satisfy*

$$\frac{d_2^{A\prime}(R^*; \boldsymbol{\omega}^A)}{d_1^{A\prime}(R^*; \boldsymbol{\omega}^A)} \left[\frac{d_2^{A\prime}(R^*; \boldsymbol{\omega}^A)}{d_1^{A\prime}(R^*; \boldsymbol{\omega}^A)} - \frac{d_2^{B\prime}(R^*; \boldsymbol{\omega}^B)}{d_1^{B\prime}(R^*; \boldsymbol{\omega}^B)} \right] \quad > \quad 0. \tag{3}$$

The relevance of Proposition 5.3 is twofold. First, if an equilibrium allocation does not satisfy (3), then there must exist at least another competitive equilibrium (as shown in Section 5.3). Second, if it is possible to guarantee –under further assumptions– that every equilibrium allocation in an economy must satisfy (3), then these additional assumptions will become sufficient conditions for a unique competitive equilibrium.[27] However, in other cases it could be a hard task to find such assumptions satisfying (3) for every equilibrium, so the following result presents a stronger condition for (3) to hold.

Lemma 5.4. *Consider the economy delineated in Proposition 5.3. If the condition*

$$\frac{\Psi_1^A(R; \boldsymbol{\omega}^A)}{d_1^A(R; \boldsymbol{\omega}^A)} \frac{\Psi_2^B(R; \boldsymbol{\omega}^B)}{d_2^B(R; \boldsymbol{\omega}^B)} - \frac{\Psi_2^A(R; \boldsymbol{\omega}^A)}{d_2^A(R; \boldsymbol{\omega}^A)} \frac{\Psi_1^B(R; \boldsymbol{\omega}^B)}{d_1^B(R; \boldsymbol{\omega}^B)} \geq 0$$

is satisfied for every competitive equilibrium with an equilibrium relative price $R \in \widehat{\mathcal{P}}$*, then condition (3) holds.*

The condition is found after substituting the slope of the offer curves (1) into (3). Again, if it is possible to guarantee –under further assumptions– that every equilibrium allocation in an economy must satisfy the relationship between the values of the agents' absolute risk aversions in Lemma 5.4, then these additional assumptions will become sufficient conditions for a unique competitive equilibrium.[28]

5.2 Case II. Uniqueness with normal offer curves

Our second main result, proved in the Appendix, presents sufficient conditions for a unique competitive equilibrium in the case that both offer curves are $l-$*normal*, but of different type.

Proposition 5.5. *Uniqueness for opposite,* $l-$*normal offer curves.* *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$*, satisfying Assumption 1. Suppose, without any loss of generality, that agent A's offer curve* $\mathcal{C}^A(R; \boldsymbol{\omega}^A)$ *is* $l = 2-$*normal with the non-degenerate* e_1-*critical allocation* $d^A(\widehat{R}^A; \boldsymbol{\omega}^A)$*, while agent B's offer curve* $\mathcal{C}^B(R; \boldsymbol{\omega}^B)$ *is* $l = 1-$*normal with the non-degenerate* e_2-*critical allocation* $d^B(\widetilde{R}^B; \boldsymbol{\omega}^B)$*, with* $\widehat{R}^A$*,* $\widetilde{R}^B \in \widehat{\mathcal{P}}$*. If*

$$d_1^A(\widehat{R}^A; \boldsymbol{\omega}^A) + d_1^B(\widetilde{R}^B; \boldsymbol{\omega}^B) \quad \geq \quad \boldsymbol{\omega}_1, \text{ or} \tag{4}$$

$$d_2^A(\widehat{R}^A; \boldsymbol{\omega}^A) + d_2^B(\widetilde{R}^B; \boldsymbol{\omega}^B) \quad \geq \quad \boldsymbol{\omega}_2, \tag{5}$$

are satisfied, then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price $R^* \in \widehat{\mathcal{P}}$*.*

[27]For instance, if we additionally assume that agent A's and B's offer curve satisfy the *gross substitute* property, the slopes of both offer curves are negative at any allocation –including the equilibrium one–, and the (negative) slope of $\mathcal{C}^A(R^*; \boldsymbol{\omega}^A)$ will be greater than that of $\mathcal{C}^B(R^*; \boldsymbol{\omega}^B)$ at any equilibrium allocation with a relative price R^*. Hence, condition (3) is satisfied and, under the additional assumption, the equilibrium is unique (Corollary 5.2.(i)).

[28]For instance, if the agents' utility function have constant relative risk-aversion as in the CES economies (e.g., $\Psi_l^A = \Psi_l^B$ with $l = 1$ and 2), one can find bounds on the initial distribution of endowments guaranteeing that any equilibrium satisfies the condition in Lemma 5.4. We will turn back to this issue for particular parametrizations of the utility function (Section 6).

The intuition of this result can be found in Figure 3(c). Given agent-A offer curve's non-degenerate e_1-critical allocation $d^A(\widehat{R}^A;\boldsymbol{\omega}^A)$, then agent-$B$ offer curve's non-degenerate e_2-critical allocation $d^B(\widetilde{R}^B;\boldsymbol{\omega}^B)$ must be located in one of the following three areas within the Edgeworth box to satisfy the conditions in Proposition 5.5. (a) If located in the northwest area of the allocation $d^A(\widehat{R}^A;\boldsymbol{\omega}^A)$ –condition (4) holds–, then both offer curves intersect once at an equilibrium allocation located in the northeast area. At the equilibrium allocation commodities are *gross complements* for agent A, while they are *gross substitutes* for agent B. (b) If $d^B(\widetilde{R}^B;\boldsymbol{\omega}^B)$ is located in the southwest area of $d^A(\widehat{R}^A;\boldsymbol{\omega}^A)$ –conditions (4)-(5) hold–, then both offer curves intersect once at the southeast area, and commodities are *gross substitutes* for both agents at the equilibrium allocation. (c) If $d^B(\widetilde{R}^B;\boldsymbol{\omega}^B)$ is located in the southeast area of $d^A(\widehat{R}^A;\boldsymbol{\omega}^A)$ –condition (5) holds–, and then both offer curves intersect once in the southeast area, and commodities are either *gross substitutes* for both agents or *gross substitutes* for agent A and *gross complements* for agent B at the equilibrium allocation. In summary, the necessary condition for uniqueness is that commodities be *gross substitutes* for at least one agent.

As a particular case of Proposition 5.5, uniqueness will be guaranteed in opposite, normal offer curves whenever the agents' offer curve, $\mathcal{C}^A(R;\boldsymbol{\omega}^A)$ and $\mathcal{C}^B(R;\boldsymbol{\omega}^B)$, intersect at the respective non-degenerate critical allocations, i.e. $\widehat{R}^A = \widetilde{R}^B$. This entails that $\widehat{R}^A \geq \widetilde{R}^B$ is a sufficient condition for uniqueness when both offer curves are opposite normal. The proof is straightforward from Theorem 5.1.

Corollary 5.6. ***Uniqueness for opposite, $l-$normal offer curves: gross substitute arc segments at a subset of the trading price set.*** *Consider a two-commodity, two-agent economy $\varepsilon = \{(\mathcal{X},\boldsymbol{\omega}^i,U^i)\}_{i=A,B}$, satisfying Assumption 1. If $\widehat{R}^A \geq \widetilde{R}^B$, then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price $R^* \in \widehat{\mathcal{P}}$.*

Note that Proposition 5.5 and Corollary 5.6 embody informal intuitions concerning uniqueness: a unique competitive equilibrium is more likely in models that either generate a large volume of trade (see Jones 1970, and Mas-Colell 1991, p.285) *or* a small volume of trade (Balasko 1995, p.97).

Regarding existing results in the literature, it is interesting to highlight that Proposition 5.5 stems from the individual demand functions and cannot be deduced from the aggregate excess demand function, even if the trading price set is restricted to $\widehat{\mathcal{P}}$. Yet, Proposition 5.5 allows us to profile constraints on aggregate excess demand functions that guarantee uniqueness of competitive equilibrium. Observe first that condition (4) –which is satisfied in the west regions of Figure 3(c)– entails a non-negative excess demand for commodity $l = 1$, $z_1(R;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B) \geq 0$, in the range $R \in [R_{\boldsymbol{\omega}^A}, \widetilde{R}^B]$. Then, by the continuity of the aggregate excess demand function, a unique root must exist for a relative price R^* satisfying $R^* \geq \widetilde{R}^B$; that is, the resulting competitive allocation is located at the subarc $\mathcal{C}^B(R,\omega^B)$ with $R \in [\widetilde{R}^B, R_{\omega^B}]$ that satisfies the *gross substitute* property. Also, condition (5) –which is satisfied in the south regions of Figure 3(c)– entails a non-negative excess demand for commodity $l = 2$ –and, thus, a non-positive excess demand for commodity $l = 1$, $z_1(R;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B) \leq 0$– in the range $R \in [\widehat{R}^A, R_{\boldsymbol{\omega}^B}]$. Once again, a unique root satisfying $R^* \leq \widehat{R}^A$ must exist, and the resulting competitive allocation is located at the subarc segment $\mathcal{C}^A(R,\omega^A)$ for $R \in [R_{\omega^A}, \widehat{R}^A]$ that satisfies the *gross substitute* property. Next, we summarize these intuitions in terms of the aggregate excess demand function.

Lemma 5.7. ***Uniqueness, opposite $l-$normal offer curves and the aggregate excess demand function.*** *Consider a two-commodity, two-agent economy $\varepsilon = \{(\mathcal{X},\boldsymbol{\omega}^i,U^i)\}_{i=A,B}$, satisfying Assumption 1.*
(i) If $z_1(R;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B) \geq 0$ for all $R \leq \widetilde{R}^B$, then there can be at most one competitive equilibrium

allocation and a unique equilibrium relative price $R^* \geq \widetilde{R}^B$.
(ii) If $z_1(R; \boldsymbol{\omega}^A, \boldsymbol{\omega}^B) \leq 0$ *for all* $R \geq \widehat{R}^A$*, then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price* $R^* \leq \widehat{R}^A$.

To conclude the study of the conditions of uniqueness for *gross substitute* and *normal* offer curves (Cases I-II), we present three particular cases. In Corollary 5.8, uniqueness arises in a degenerate trading (relative) price set at Theorem 5.1 and Corollary 5.6, because the offer curve is locally convex around the autarkic allocation (Proposition 3.5.(*i*)).

Corollary 5.8. *(Mas-Colell et al. 1995, Prop.17.F.5)* ***Unique non-trading equilibrium.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$. *If* $\widehat{\mathcal{P}} = \{R_{\boldsymbol{\omega}}\}$ *with* $R_{\boldsymbol{\omega}} = R_{\boldsymbol{\omega}^A} = R_{\boldsymbol{\omega}^B}$*, then there exits a unique competitive equilibrium allocation* $\mathbf{x}^{i*} = \boldsymbol{\omega}^i$ *with* $i = A,\ B$*, and a unique equilibrium relative price* $R^* = R_{\boldsymbol{\omega}}$.

In Corollary 5.9 we present conditions that guarantee uniqueness in the case that both agents' preferences are homothetic. The proof is a straightforward result from Proposition 4.4.(*ii*), Corollary 5.2 and Proposition 5.5.

Corollary 5.9. ***Uniqueness with homothetic preferences.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$*, satisfying Assumption 1. Suppose that preferences are homothetic for both agents, and that one of the following conditions hold: (a) at least one agent's offer curve satisfies the gross substitute property; or, (b) conditions (4) or (5) are satisfied. Then, there can be at most one competitive equilibrium allocation and a unique equilibrium relative price* $R^* \in \widehat{\mathcal{P}}$.

In Corollary 5.10 we present conditions in the case of additively separable preferences. We mainly focus on the properties of the absolute and the relative risk aversion evaluated at the optimal allocations. Part (*a*) is straightforward from Proposition 5.5. Part (*b*) restricts the value of the relative risk aversion for at least one agent so that the resulting offer curve satisfies the *gross substitute* property (according to Lemma 4.5), and, consequently, competitive equilibrium is unique (Corollary 5.2). Parts (*c*) and (*d*), proved in the Appendix, restricts the the distributions of initial endowments and preferences guaranteeing that every equilibrium allocation in an economy must satisfy the relationship in Lemma 5.4. Specifically, Part (*c*) confines the autarkic allocation to be placed in the northwest region of the Edgeworth box, together with requiring a non-decreasing profile for the absolute risk aversions as the relative price increases.[29] And, Part (*d*) assumes a constant, positive relative risk aversion, together with a condition that relates the initial distribution of endowments and the value of utility parameters. This condition delineates a convex, increasing curve within the Edgeworth box, and it has a graphical interpretation. For the simplest constant relative risk aversion ($\Psi^i_l(R; \boldsymbol{\omega}^i) = \Psi > 0$) the condition becomes $\omega^A_2/\omega^A_1 \geq \boldsymbol{\omega}_2/\boldsymbol{\omega}_1$, so that if agent A's endowment is located above or on the diagonal of the Edgworth box (and Assumption 1 holds), then competitive equilibrium is unique.[30] Once again, this illustrates that the informal intuitions concerning unique competitive equilibrium is more likely in models generating a small volume of trade (Balasko 1995, p.97).

[29] These conditions, however, restricts the torsion of a regular curve –i.e., the second curvature– to uniquely determine offer curves. As indicated in Section 3.3 and in the comments to Lemma 4.2, we are quite reluctant to present assumptions out of the range of standard preference theory involving cardinal welfare considerations. Yet, assumptions on the profile of the absolute relative risk aversion ($\Psi^i_l(R; \boldsymbol{\omega}^i)/d^i_l(R; \boldsymbol{\omega}^i)$) are usual in financial markets literature (see, e.g., Hens et al. 1995).

[30] Geanokoplos et al. (2018, Prop.4) only considers the particular distribution of endowments located *on* the diagonal of the Edgeworth box.

Corollary 5.10. ***Uniqueness with additively separable preferences.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$*, satisfying Assumption 1. Suppose that preferences are additively separable for both agents, and that one of the following conditions hold: (a) conditions (4) or (5) are satisfied; (b) for at least one agent* $\Psi_l^i(R; \boldsymbol{\omega}^i) \in [0, 1]$ *is satisfied for all* $R \in \mathcal{P}$ *and any* $l = 1, 2$*; (c) (Geanokoplos et al. 2018, Propositions 2 and 5) the absolute risk aversion for all commodities is non-decreasing –i.e.* $\partial[\Psi_l^i(R; \boldsymbol{\omega}^i)/d_l^i(R; \boldsymbol{\omega}^i)]/\partial R < 0$ *for all* $R \in \widehat{\mathcal{P}}_{\boldsymbol{\omega}^i}$*, for all* $l = 1, 2$ *and* $i = A, B$*–, and the endowment distribution satisfies* $\omega_1^A \leq \boldsymbol{\omega}_1/2 \leq \omega_1^B$ *and* $\omega_2^A \geq \boldsymbol{\omega}_2/2 \geq \omega_2^B$*; or, (d) the relative risk aversion takes a positive constant –i.e.* $\Psi_l^i(R; \boldsymbol{\omega}^i) = \Psi_l^i > 0$ *for all* $R \in \widehat{\mathcal{P}}_{\boldsymbol{\omega}^i}$*, for any* $l = 1, 2$ *and* $i = A, B$*–, and the endowment distribution satisfies*

$$\frac{\omega_2^A}{\omega_1^A} \geq \frac{\Psi_2^A \Psi_1^B}{\Psi_1^A \Psi_2^B + [\Psi_2^A \Psi_1^B - \Psi_1^A \Psi_2^B]\omega_1^A} \frac{\boldsymbol{\omega}_2}{\boldsymbol{\omega}_1}.$$

Then, there can be at most one competitive equilibrium allocation and a unique equilibrium relative price $R^* \in \widehat{\mathcal{P}}$*.*

5.3 Multiple equilibria with opposite, normal offer curves

In this section we provide a sufficient condition for the existence of multiple equilibria when the offer curves are opposite normal. A consequence of the intuitions of Proposition 5.5, provided by Figure 3(c), is that multiple equilibrium allocations is possible whenever there exists a competitive equilibrium satisfying that all commodities are simultaneously *gross complements* for all agents– i.e., a positive slope of all offer curves at such competitive equilibrium. This is not, however, a sufficient condition for the existence of multiple equilibria, since Proposition 5.3 must not hold. As the following result shows, further restrictions on the profile of the agents' offer curve are required for multiple equilibrium allocations to exist; specifically, a rise in the relative price of commodity $l = 1$ increases agent A's demand for this commodity by more than it decreases agent B's demand[31] –i.e., the slope of agent B's offer curve must be steeper than that of agent A's.

Proposition 5.11. ***Normal offer curves and multiple equilibria. Full characterization.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$ *satisfying Assumption 1. Consider that the agents' offer curve are l-monotones, and that there exists an equilibrium allocation for the equilibrium price ratio* R^**. If* (*a*) *agent A's and B's offer curve satisfy the gross complement property at a boundary of the equilibrium allocation, and* (*b*) *the absolute elasticity of substitution for agent B is higher than the one for agent A at the equilibrium allocation,*

$$0 < \frac{d_2^{A\prime}(R^*; \boldsymbol{\omega}^A)}{d_1^{A\prime}(R^*; \boldsymbol{\omega}^A)} < \frac{d_2^{B\prime}(R^*; \boldsymbol{\omega}^B)}{d_1^{B\prime}(R^*; \boldsymbol{\omega}^B)}, \tag{6}$$

then the equilibrium allocation is not unique.

This result is related to the Index Theorem (Kehoe 1998, Sec.3.2 and Mas-Colell et al. 1995, Prop.17.D.2). In terms of our setting, it would be enough to check that the excess demand function for commodity $l = 1$ is positively sloped at some particular equilibrium (relative) price R^* to guarantee that there exist multiple equilibria. Thus, a testable condition to check the existence of

[31] In terms of the literature on international economics –which considers commodities $l = 1$ and $l = 2$ as country A's and country B's home good, respectively–, "the wealth effects in each country are so biased toward the home commodity that an increase in the [(relative) price R,] actually increases the demand for [commodity $l = 1$] in country 1 by more than it decreases the demand from country 2." (Mas-Colell et al. 1995, Ex.17.F.3, p.615).

multiple equilibria becomes $z_1'(R^*;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B) = d_1^{A\prime}(R^*;\boldsymbol{\omega}^A)+d_1^{B\prime}(R^*;\boldsymbol{\omega}^A) > 0$. Following this idea, Toda et al. (2017, Prop.4) provide a condition to guarantee the existence of multiple equilibria in economies with symmetric, general additively separable preferences and symmetric endowments. However, even though checking whether a competitive equilibrium with the relative price R^* satisfies $z_1'(R^*;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B) > 0$ is a sufficient condition for the existence of multiple equilibria, this condition alone *cannot* be used to provide a full characterization of the region of parameters resulting in multiple competitive equilibria in *opposite, normal* offer curves.[32] Thus, Proposition 5.11 is of great interest as it provides explicit and testable conditions, as will be shown useful in the examples studied in Section 6.

5.4 Case III. Uniqueness with normal and non-normal offer curves

Our third main result provides sufficient conditions for a competitive equilibrium to be unique in the case that one of the offer curves is *non-normal* while the other is *normal*.

Proposition 5.12. ***Uniqueness for normal and non-normal offer curves.*** *Consider a two-commodity, two-agent economy $\varepsilon = \{(\mathcal{X},\boldsymbol{\omega}^i,U^i)\}_{i=A,B}$ satisfying Assumption 1. Suppose, with any loss of generality, that agent A's offer curve $\mathcal{C}^A(R;\boldsymbol{\omega}^A)$ is non-normal displaying three non-degenerate critical allocations satisfying $d_1^{A\prime}(\widehat{R}_1^A;\boldsymbol{\omega}^A)=0$, $d_2^{A\prime}(\widehat{R}_2^A;\boldsymbol{\omega}^A)=0$ and $d_2^{A\prime}(\widehat{R}_3^A;\boldsymbol{\omega}^A)=0$ with $\widehat{R}_1^A < \widehat{R}_2^A < \widehat{R}_3^A$, while agent B's offer curve $\mathcal{C}^B(R;\boldsymbol{\omega}^B)$ is $l=1-$normal with the non-degenerate e_2-critical allocation $d^B(\widetilde{R}^B;\boldsymbol{\omega}^B)$ with $\widetilde{R}^B \in \underline{\mathcal{P}}_{\boldsymbol{\omega}^B}$. If*

$$d_2^A(\widehat{R}_3^A;\boldsymbol{\omega}^A) + d_2^B(\widetilde{R}^B;\boldsymbol{\omega}^B) < \boldsymbol{\omega}_2, \tag{7}$$

then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price $R^* \in (R_{\boldsymbol{\omega}^A},\widehat{R}_2^A)$.

The intuition of this result can be found in Figure 3(d). Given agent-A offer curve's non-degenerate e_2-critical allocation $d^A(\widehat{R}_3^A;\boldsymbol{\omega}^A)$, then agent-$B$ offer curve's non-degenerate e_2-critical allocation $d^B(\widetilde{R}^B;\boldsymbol{\omega}^B)$ must be located at the south area of the allocation $d^A(\widehat{R}_3^A;\boldsymbol{\omega}^A)$ to satisfy the conditions in Proposition 5.12. Then both offer curves intersect once at an equilibrium allocation. Once again, Proposition 5.12 also illustrates the informal intuition concerning a unique competitive equilibrium that is more likely in models generating small volume of trade (Balasko 1995, p.97).

Finally, with regard existing results in the literature, Proposition 5.12 stems from the individual demand functions and cannot be deducted from the aggregate excess demand function. Yet, Proposition 5.12 allows us to profile constraints on aggregate excess demand functions that guarantee uniqueness of competitive equilibrium in the presence of *normal* and *non-normal* offer curves. Specifically, Lemma 5.7.(i) holds in the economy described in Proposition 5.12, and Lemma 5.7.(ii) can be restated as follows: if $z_1(R;\boldsymbol{\omega}_1^A,\boldsymbol{\omega}^B) \leq 0$ is satisfied for all $R \geq \widehat{R}_1^A$, then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price $R^* \leq \widehat{R}_1^A$.

6 Examples

In this section, we illustrate our results with well-known examples presented in the literature of multiple equilibria. First, we study a Shapley-Shubik economy; then, we focus on CES economies,

[32] In our setting, a full characterization would require studying the equations $z_1'(R^*;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B) > 0$ and $z_2'(R^*;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B) < 0$ simultaneously, in addition to checking $d_l^{i\prime}(R^*;\boldsymbol{\omega}^i) > 0$ for $l = 1$, 2 and $i = A$, B. This joint analysis would obtain the same set of parameters resulting from Proposition 5.11.

with ‘convenient’ functional forms widely adopted in applied models (see Shoven and Whalley 1992). In both cases, we study the case of a particular distribution of preferences and endowments denoted as “supersymmetry” (i.e. the agents’ preferences and endowments are mirror images of one another).

Example 1. A Shapley-Shubik economy with power function utilities. (Mas-Colell 1991, Ex.1; Mas-Colell et al. 1995, Ex.15.B.2; and, Bergstrom et al. 2009, Secs.5.2 and 6.1.) Consider the non-homothetic preferences represented by a quasilinear utility function $U^i(x_1, x_2) = ax_1 + u(x_2)$ with $a > 0$ (see Bergstrom et al. 2009), such that the non-linear element of the utility function is a power function of the form $u(x_2) = -x_2^{-b}/b$ with $b > 0$. This representation is a quasi-concave, continuous utility function with a constant and non-decreasing relative risk aversion $\Psi^i(R; \boldsymbol{\omega}^i) = (0, 1+b) > (0, 1)$ for all $R \in \mathcal{P}$. Initially, we show that the corresponding offer curve is *normal* for these preferences and any endowment (see Figure 2(b)). The proof is in the Appendix.

Proposition 6.1. *Consider the utility function* $U(x_1, x_2) = ax_1 - x_2^{-b}/b$ *[resp.,* $U(x_1, x_2) = -x_1^{-b}/b + ax_2$ *] with* a, $b > 0$. *Then, (i) the offer curve is* $l = 2-$*normal [resp.,* $l = 1-$*normal]; (ii) there exists a unique non-degenerate* e_1-*critical allocation at the relative price* $\widehat{R} = a\left[\frac{b+1}{b}\omega_2^i\right]^{b+1} > R_{\boldsymbol{\omega}^i}$ *[resp., a unique non-degenerate* e_2-*critical allocation at* $\widetilde{R} = 1/\left[a\left[\frac{b+1}{b}\omega_1^i\right]^{b+1}\right] < R_{\boldsymbol{\omega}^i}$*]; and, (iii) there exists an inflection point at the relative price* $\breve{R} = a\left[\frac{b+2}{b}\omega_2^i\right]^{b+1} > \widehat{R}$ *[resp.,* $\breve{R} = 1/\left[a\left[\frac{b+2}{b}\omega_1^i\right]^{b+1}\right] < \widetilde{R}$*].*

Next, we present conditions that guarantee uniqueness of competitive equilibrium in our particular Shapley-Shubik economies based on the general findings proved in Section 5. In our first result, all agents’ preferences are represented by a similar quasilinear utility function with power function utility. The equilibrium is unique directly from Corollary 5.2.(*ii*), as $\overline{C}^A(R; \boldsymbol{\omega}^A)$ is e_2-monotone while $\underline{C}^B(R; \boldsymbol{\omega}^B)$ satisfies the *gross substitute* property (or vice versa).

Lemma 6.2. ***Uniqueness with similar quasilinear utility functions with power function utility.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$ *satisfying Assumption 1. If both agents’ preferences are represented by similar Shapley-Shubik utility functions with the same profile, i.e.* $U^i(x_1^i, x_2^i) = a^i x_1^i - \left(x_2^i\right)^{-b^i}/b^i$ *[or,* $U^i(x_1^i, x_2^i) = -\left(x_1^i\right)^{-b^i}/b^i + a^i x_2^i$*] for* $i = A$, B, *then there exists a unique equilibrium.*

In our second result, we consider an Shapley-Shubik economy with mirror quasilinear utility functions with power function utilities, so that the arc segments $\overline{C}^A(R; \boldsymbol{\omega}^A)$ and $\overline{C}^B(R; \boldsymbol{\omega}^B)$ are $l = 2-$ and $l = 1-$*normal* respectively. In this setting, we can restrict the distribution of endowments and the parameters of preferences by assuming conditions (4)-(5) in Proposition 5.5 hold, to guarantee the existence of a unique competitive equilibrium.

Lemma 6.3. ***Uniqueness for opposite,*** $l-$***normal offer curves in a mirror Shapley-Shubik economy.*** *Consider a two-commodity, two-agent economy* $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$, *and, without any loss of generality, suppose Assumption 1 holds. Assume also that agent A’s utility function is* $U^A(x_1^A, x_2^A) = a^A x_1^A - \left(x_2^A\right)^{-b^A}/b^A$, *while agent B’s is* $U^B(x_1^B, x_2^B) = -\left(x_1^B\right)^{-b^B}/b^B + a^B x_2^B$. *If*

$$\left(\omega_2^A\right)^{b^A}\omega_1^B \geq \Omega_1, \text{ or, } \omega_2^A\left(\omega_1^B\right)^{b^B} \geq \Omega_2$$

with $\Omega_1 \equiv [\left(b^A\right)^{b^A} b^B]/[\left(b^A+1\right)^{b^A+1} a^A]$ *and* $\Omega_2 \equiv [\left(b^B\right)^{b^B} b^A]/[\left(b^B+1\right)^{b^B+1} a^B]$, *then there can be at most one competitive equilibrium allocation, and a unique equilibrium relative price* $R^* \in \widehat{\mathcal{P}}$.

Interestingly, the condition for uniqueness in Corollary 5.6 –i.e., $\widehat{R}^A \geq \widetilde{R}^B$–, in our mirror Shapley-Shubik economy becomes $\left(\omega_2^A\right)^{b^A+1}\left(\omega_1^B\right)^{b^B+1} \geq \Omega_1\Omega_2$, hence becoming a particular case of Lemma 6.3. Finally, our third result considers a further simplification of Lemma 6.3 by considering a mirror endowment distribution and symmetric utility parameters. In this setting it is easy to show that $R^* = 1$ is always an equilibrium price ratio, though it is not necessarily unique (see Bergstrom et al. 2009, Secs.5.2 and 6.1).

Corollary 6.4. ***Uniqueness in a supersymmetric, mirror Shapley-Shubik economy.*** *Consider the economy defined in Lemma 6.3. Assume that endowments are given by $\boldsymbol{\omega}^A = (\omega, \delta)$ and $\boldsymbol{\omega}^B = (\delta, \omega)$, and the utility parameters are $a^A = a^B = a$, and $b^A = b^B = b$. Due to Assumption 1, parameters are restricted to hold $1/a^{1/(b+1)} \geq \delta$. If*

$$\frac{1}{a^{1/(b+1)}} \geq \delta \geq \frac{b}{b+1}\frac{1}{a^{1/(b+1)}}, \tag{8}$$

then there can be at most one (symmetric) competitive equilibrium allocation, and a unique equilibrium relative price $R^* = 1$.

In the literature of international trade, it is usually assumed that $\omega > \delta$, suggesting that countries have a greater supply of the *domestic good*. The case of (production) specialization is represented by $\delta = 0$, while a country is diversified in (production of) commodities if $\delta > 0$. The Corollary shows that the more diversified a country is, the greater the likelihood of a unique equilibrium. The condition (8) also entails that $\widehat{R}^A \geq 1 \geq \widetilde{R}^B$, so Corollary 5.6 applies and then a unique equilibrium relative price exists. Finally, observe that Mas-Colell et al. (1995, Ex.15.B), which considers $a = 1$, $b = 8$, and $(\omega, \delta) = (2, r)$, does not satisfy (8) since it is assumed that $r = 2^{8/9} - 2^{1/9} < 8/9$. Otherwise, for any $r \geq 8/9$ a unique equilibrium will be found.

We conclude this example by addressing multiplicity of equilibria. Recall that mirror supersymmetry guarantees that $R^* = 1$ is always an equilibrium relative price of the symmetric competitive equilibrium. Proposition 5.11 states that there exists multiple equilibria provided commodities are *gross complements* at the equilibrium allocation with R^* and, *in addition*, the positive slope of agent-A's offer curve is lower than that of agent B at such an equilibrium allocation. Interestingly, for this mirror supersymmetric Shapley-Shubik economy with opposite, normal offer curves, the competitive equilibrium at $R^* = 1$ is also key to characterize uniqueness, because in the case that any of these two conditions do not hold at the symmetric equilibrium allocation, then competitive equilibrium is not unique (Proposition 5.11). This allows us to provide a full characterization of the set of parameters that guarantee uniqueness and the existence of multiple equilibria for this particular case. The Proposition, proved in the Appendix, is an extension of Toda et al. (2017, Prop.3), who assume that $\omega = \delta$.

Proposition 6.5. ***Multiple equilibria in the mirror, supersymmetric Shapley-Shubik economy: full characterization.*** *Consider the supersymmetric Shapley-Shubik economy defined in Corollary 6.4, which presents a (symmetric) competitive equilibrium with the equilibrium relative price $R^* = 1$. There exists another (asymmetric) competitive equilibrium allocation, besides the (symmetric) equilibrium if and only if $\rho < 0$ and*

$$\delta < \frac{b-1}{b+1}\frac{1}{a^{1/(b+1)}}.$$

Otherwise, there can be at most one competitive equilibrium, i.e. the symmetric one with $R^* = 1$.

As a straightforward consequence, $\delta = 0$ –a case denoted to as *full specialization* in the international trade literature– is a sufficient condition for the existence of multiple equilibria.[33] Note also that the condition in Proposition 6.5 entails that, for multiple equilibria to exist, the relative prices that characterize each agent's non-degenerate critical allocations must fall outside the range $[(1-1/b)^{b+1}, (1+1/b)^{-(b+1)}]$; specifically, $\widehat{R}^A < (1-1/b)^{b+1}$ and $\widetilde{R}^B > (1-1/b)^{-(b+1)}$, with $\widetilde{R}^B = [\widehat{R}^A]^{-1}$ because of supersymmetry. To conclude, observe that Proposition 6.5 provides a counterexample to Hildenbrand (1983, Prop.1), which states that collinearity of individual endowments –in this supersymmetric case, $\delta = \omega$– is sufficient to guarantee a unique equilibrium relative price. Hildenbrand's result does not hold here because it crucially relies on the assumption that there exits a continuum of consumers. One might think that using our result it would make it possible to construct a sequence of economies converging into a continuum economy of the Hildenbrand type but with several equilibria. This is not possible because, as indicated by Kirman et al. (1986, p.460), "to construct the sequence we need to add at least a fixed amount to the total resources for each consumer added. Hence mean endowment and hence mean income do not converge and we cannot satisfy Hildenbrand's finite mean demand condition." (p.460)□

Example 2. A CES economy. (Kehoe 1991, Ex.2.1, and 1998, Ex.1; Mas-Colell et al. 1995, Exerc.15.B.6; Gjerstad 1996; Qin et al. 2009, Ex.2; Chipman 2009; and, Toda et al. 2017, Sec.2.) Consider the homothetic preferences represented by the constant-relative-risk-aversion and constant-elasticity-of-substitution (CES) utility function $U(x_1, x_2) = a\,x_1^\rho + b\,x_2^\rho$ with a, $b > 0$ and $\rho \leq 1$. This is a quasi-concave continuous utility function with the constant Hicks-Allen elasticity of substitution $\sigma = 1/(1-\rho)$ and the constant relative risk aversion $\Psi_l(R;\omega) = 1-\rho$ for all $R \in \mathcal{P}$ with $l = 1$, 2. Initially, we show that the corresponding offer curve is *normal* for these preferences and any endowment (see Figure 2(c)). The proof is in the Appendix.

Proposition 6.6. Characterization of the offer curve for a CES utility function. *Consider a* CES *utility function.*
(i) If $\rho \in [0,1]$, then the offer curve $C(R;\boldsymbol{\omega})$ satisfies the gross substitute property for all $R \in \mathcal{P}$, and there exist no critical allocations; and,
(ii) If $\rho < 0$, then (ii.1) the two complement arc segments $\underline{C}(R;\boldsymbol{\omega})$ and $\overline{C}(R;\boldsymbol{\omega})$ are $l = 1-$ and $l = 2-$normal, respectively, and (ii.2) there exist only two critical allocations for the relative prices $\widehat{R} \in (R_\omega, +\infty)$ and $\widetilde{R} \in (0, R_\omega)$ satisfying $d_1'(\widehat{R};\boldsymbol{\omega}) = 0$ and $d_2'(\widetilde{R} : \boldsymbol{\omega}) = 0$, respectively.

Next, we present particular conditions –derived from those general findings proved in Section 5–, that guarantee uniqueness of competitive equilibrium in *CES economies*, i.e., in a two-commodity, two-agent economy with agents' preferences represented by a CES utility function. Observe that, as a first straightforward result, if $\rho^i \in [0,1]$ for some agent i –so that the agent i's offer curve satisfies the *gross complement* property (Proposition 6.6.(i))–, then there exists a unique competitive equilibrium (Theorem 5.1 and Corollary 5.2).

In our second result, both agents' preferences are represented by a CES utility function with $\rho^i < 0$ with $i = A$, B, so that the arc segments $\overline{C}^A(R;\boldsymbol{\omega}^A)$ and $\overline{C}^B(R;\boldsymbol{\omega}^B)$ are $l = 2-$ and $l = 1-$*normal*, respectively. In this setting, we restrict the distribution of endowments and the parameters of preferences to guarantee the existence of a unique competitive equilibrium. In Part (a), the distribution of endowments are restricted so that initial endowments are placed above

[33] In this case, the non-degenerate relative prices become $\widetilde{R}^A = 0$ and $\widetilde{R}^B = \infty$. Since the offer curve exhibits a positive slope at the symmetric competitive equilibrium with $R^* = 1$, to prove the existence of an additional asymmetric equilibrium it is only left to show that the slope of agent-A's offer curve at R*=1 is steeper than the slope of agent-B's offer curve (Proposition 5.11). Interestingly, these asymmetric equilibria are likely to be corner solutions (see, e.g., Bergstrom et al. 2009, Ex.5).

the diagonal of the Edgeworth box. Since both agents' preferences are additively separable and exhibit a constant and positive relative risk aversion, Corollary 5.10.(d) directly applies. In Part (b), the distribution of endowments and the value of utility parameters are assumed to satisfy conditions (4)-(5) in Proposition 5.5. Thus, the agent-B offer curve's critical allocation are placed at the west or the south of the agent-A offer curve's critical allocation (Figure 3(c)).

Lemma 6.7. ***Uniqueness for opposite, $l-$normal offer curves in a CES economy.*** *Consider a two-commodity, two-agent economy $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$ satisfying Assumption 1. Suppose that preferences are represented by the* CES *utility function $U^i(x_1, x_2) = a^i\, x_1^{\rho^i} + b^i\, x_2^{\rho^i}$ for $i = A,\ B$, and let $\widehat{R}^A$ and $\widetilde{R}^B$ be the only ratio of prices satisfying $d_1^{A\prime}(\widehat{R}^A; \boldsymbol{\omega}^A) = 0$ and $d_2^{B\prime}(\widetilde{R}^B; \boldsymbol{\omega}^B) = 0$, respectively. Suppose that $\rho^i < 0$ is satisfied by both agents, and that one of the following conditions holds: (a) $\omega_2^A/\omega_1^A \geq \boldsymbol{\omega}_2/\boldsymbol{\omega}_1$; or, (b) conditions*

$$
\begin{aligned}
\frac{1-\rho^A}{-\rho^A}\left(\frac{b^A}{a^A}\widehat{R}^A\right)^{1/(\rho^A-1)}\omega_2^A &\geq \omega_1^A + \frac{1}{\rho^B}\omega_1^B;\ \textit{or}, \\
\frac{1-\rho^B}{-\rho^B}\left(\frac{b^B}{a^B}\widetilde{R}^B\right)^{-1/(\rho^B-1)}\omega_1^B &\geq \frac{1}{\rho^A}\omega_2^A + \omega_2^B,
\end{aligned}
$$

are satisfied. Then, there can be at most one competitive equilibrium allocation and a unique equilibrium relative price $R^* \in \widehat{\mathcal{P}}$.

Our third result considers a further simplification of Lemma 6.7 by considering a mirror endowment distribution and utility parameters (see Kehoe 1991, Ex.2.1, and 1998, Ex.1; Mas-Colell et al. 1995; Exerc.15.B.6 and 17.D.1; Chipman 2009; and, Toda et al. 2018, Sec.2). In this setting it is easy to show that $R^* = 1$ is always an equilibrium price ratio, though it is not necessarily unique (see Chipman 2009, Sec.3 and Toda et al. 2018, Sec.2). The conditions in Corollary 6.8 are found from conditions Lemma 6.7.(i) and (ii), respectively, after making use of inequality $\widehat{R}^A \leq 1 \leq \widetilde{R}^B$ to substitute the price ratios. (Otherwise if $\widehat{R}^A \geq \widetilde{R}^B$, Corollary 5.6 applies and the equilibrium is unique.)

Corollary 6.8. ***Uniqueness in a supersymmetric, mirror* CES *economy.*** *Consider the economy delineated in Lemma 6.7. Suppose that the utility parameters are $\rho^A = \rho^B = \rho$, $a^A = b^B = \alpha^{1-\rho}$ and $b^A = a^B = \beta^{1-\rho}$, and that the endowments are given by $\boldsymbol{\omega}^A = (\omega, \delta)$ and $\boldsymbol{\omega}^B = (\delta, \omega)$. Due to Assumption 1, parameters are restricted to hold $(\alpha/\beta)^{-1} \geq \delta/\omega$. If $\rho < 0$ and*

$$
\left[\frac{\alpha}{\beta}\right]^{-1} \geq \frac{\delta}{\omega} \geq \min\left\{1, \left[\frac{1}{(-\rho)} + \frac{(1-\rho)}{(-\rho)}\frac{\alpha}{\beta}\right]^{-1}\right\},
$$

then there can be at most one competitive equilibrium allocation and a unique equilibrium relative price $R^* = 1$.

Initially, note that Chipman (2009, Th.1) proves that in the supersymmetric CES economy $\rho \geq -1$ is a sufficient condition for uniqueness for any combination of parameters. However, the condition in Corollary 6.8 only guarantees uniqueness for some parameter combinations. This is because Corollary 6.8 is a consequence of Proposition 5.5, so that the conditions given for uniqueness are sufficient, but not necessary. Indeed, Corollary 6.8 provides a region of utility and endowment parameters α/β, $\rho < 0$ and δ/ω such that the competitive equilibrium is unique. Such a region of parameters always exists, since the second term in the brackets at the right member in the expression in Corollary 6.8 is always less than $(\alpha/\beta)^{-1}$, for any relative preference of commodities. For instance, the particular combination of parameters $\alpha/\beta = 1$ and $\delta/\omega = 1$ results in the no-trading equilibrium defined in Corollary 5.8 for any $\rho < 0$.

We can provide some intuitions for Corollary 6.8 from international trade and financial economics. In terms of the international trade literature, if preferences are biased towards the *domestic good* $(\alpha/\beta > 1)$, then the condition in the Corollary identifies a region of parameters with a higher endowment in the *domestic good* $\omega > \delta > 0$ for any $\rho < 0$, so that the competitive equilibrium is unique. In terms of the financial economics literature, let us initially reconsider the economy $\varepsilon = \{(\mathcal{X}, \boldsymbol{\omega}^i, U^i)\}_{i=A,B}$ as an intertemporal economy with two commodities and two types of agents: the *impatient* agents with utility parameters $a^i/b^i > 1$, and the *patient* agents with $a^{i'}/b^{i'} < 1$. In the mirror supersymmetric economy of Corollary 6.8, if type-A agents are impatient $(\alpha/\beta > 1)$ [resp. type-B agents are patient] but transfer wealth to the future [resp. to the present] because of Assumption 1, then a necessary condition for uniqueness is that agents be wealthier in the commodity they care more about $(\omega > \delta)$. Conversely, if type-A agents are patient $(\alpha/\beta < 1)$ [resp. type-B agents are impatient], transfer wealth to the future [resp. to the present], and all agents exhibit a low risk aversion (specifically $\Psi^i = 1 - \rho < 2$ for $i = A$, B, as $\rho > -1$ is found in $\frac{1}{(-\rho)} + \frac{(1-\rho)}{(-\rho)}\frac{\alpha}{\beta} > 1$ at the lower threshold in the Corollary), then the region of endowment parameters that results in a unique equilibrium enlarges as α/β lowers to become a region that converges to the one satisfying $\delta \geq \omega$. Thus, we can summarize that uniqueness is more likely when agents are relatively more endowed with the commodity they relatively cared more about.

We conclude this example by addressing multiplicity of equilibria. Recall that mirror supersymmetry guarantees that $R^* = 1$ is always an equilibrium relative price of the symmetric competitive equilibrium. Proposition 5.11 states that there exists multiple equilibria provided commodities are *gross complements* at the equilibrium allocation with R^* and, *in addition*, the positive slope of agent-A's offer curve is lower than that of agent B at such an equilibrium allocation. If any of these two conditions do not hold at the symmetric equilibrium allocation, then competitive equilibrium is not unique (Proposition 5.11). This allows us to provide a full characterization of the set of parameters that guarantee uniqueness and the existence of multiple equilibria for this particular case. The proof is in the Appendix.

Proposition 6.9. *Multiple equilibria in the mirror, supersymmetric* CES *economy: full characterization.* *Consider the supersymmetric* CES *economy defined in Corollary 6.8, which presents a symmetric competitive equilibrium with the equilibrium relative price $R^* = 1$. There exists another asymmetric competitive equilibrium allocation, besides the symmetric equilibrium relative price $R^* = 1$, if and only if $\rho < 0$ and*

$$\frac{1}{1-\rho} \quad < \quad 1 - \frac{1}{2}\frac{\frac{\omega}{\omega+\delta} + \frac{\alpha}{\beta}\frac{\delta}{\omega+\delta}}{\frac{\frac{\alpha}{\beta}}{1+\frac{\alpha}{\beta}}}. \tag{9}$$

Otherwise, there can be at most one competitive equilibrium, i.e. the symmetric one with $R^* = 1$.

Proposition 6.9 complements Toda et al. (2017, Prop.1), as we present a full characterization both for uniqueness and for existence of multiple equilibria. Concerning uniqueness, we can find from condition (9) that $1/(1-\rho) \geq 1/2$ –i.e., $\rho \geq -1$– is a sufficient condition for uniqueness of the symmetric equilibrium, as already pointed out by Chipman (2009, Th.1). In contrast, $\rho < -1$ becomes a necessary condition for multiple equilibrium to exist (Toda et al. 2017, Remark 1). In addition, Proposition 6.9 states that if the left hand-side of (9) is less or equal to zero, then there exists only one competitive equilibrium. Then, because of Assumption 1, $\alpha/\beta < 1$ is a sufficient condition for uniqueness of the symmetric equilibrium. In contrast, the Proposition states that multiple equilibria is possible in the mirror supersymmetric CES economy only if the left hand-side of (9) is not negative; thus, because of Assumption 1, a necessary condition for the existence of multiple equilibria becomes $\alpha/\beta > 1$, as already indicated in Chipman (2009, Th.2). Intuitively, in terms of international trade, multiple equilibria is possible if the trading parties have a relative preference for their export goods; or, in term of financial economics, multiple equilibria is more likely if patient [resp., impatient] agents transfer wealth to the future [resp., present]. Finally, observe that it is easy to check that the parameters proposed in the literature of multiple equilibria satisfy condition (9) in the Proposition.[34]

Note that multiplicity of equilibria arise as the unbalance in endowments becomes greater –i.e. a low enough δ/ω– combined with a higher preference for the most endowed good –i.e. a high enough α/β.[35] Finally, as mentioned in Example 1, Proposition 6.9 together with collinear endowments –i.e., $\delta/\omega = 1$– provides another counterexample to Hildenbrand (1983, Prop.1), so the assumption of a continuum of consumers is not satisfied.□

We conclude this section by indicating that no utility function exhibiting a *non-normal* offer curve has been found in the literature, so we have been unable to check the condition in Proposition 5.12 in any example. Further comments on this issue are reported in the concluding section.

7 Concluding Comments and Extentions

In this paper, we have defied the usual criticism to the literature on uniqueness of competitive equilibrium. Namely restrictions to guarantee uniqueness are imposed on the properties in economic aggregates instead of constraining the fundamentals of the economy. Specifically, we have provided explicit and testable sufficient conditions to guarantee that a competitive equilibrium is unique by restricting the distribution of endowments and preferences. These constraints have shaped the profile of the offer curves at each side of the market in exchange economies with two commodities and two agents –or two groups of homogeneous agents or two countries.

Our simple two-agent two-commodity exchange setting seems to limit the scope of this paper. So a straightforward claim is the need to extend our approach to more (types of homogeneous) agents and more commodities (as in Kehoe et al. 1991) or to explore more general environments such as economies with production or intertemporal economies (as in Kehoe 1985, 1991). We may

[34] Some illustrations of multiple equilibria that satisfy Assumption 1 (i.e. $(\alpha/\beta)(\delta/\omega) < 1$) are (observe that all satisfied the necessary condition $\alpha/\beta > 1$): $\alpha/\beta = 1024^{1/5}$, $\delta/\omega = 1/12$ and $\rho = -4$ in Kehoe (1991, Ex.2.1 and 1998, Ex.1); $\alpha/\beta = (37/12)^{1/3}$, $\delta/\omega = 0$ and $\rho = -2$ in Mas-Colell et al. 1995 (Exerc.15.B.6); $\alpha/\beta = (70)^{1/5}$, $\delta/\omega = 0/20$ and $\rho = -4$ in Chipman (2009, Fig.3); or, $\alpha/\beta = (1024)^{1/10}$, $\delta/\omega = 1/5$ and $\rho = -9$ in Chipman (2009, Fig.6). Interestingly, the "neutral" equilibrium defined in Chipman (2009) matches with the upper threshold of (9), so a unique equilibrium exists: $\alpha/\beta = (70)^{1/5}$, $\delta/\omega = 0/20$ and $\rho = -4$ in Chipman (2009, Fig.3); or, $\alpha/\beta = [(1/7)/(1-1/7)]^{1/10}$, $\delta/\omega = 1/4$ and $\rho = -9$ in Chipman (2009, Fig.5); and $\alpha/\beta = 16^{1/4}$, $\delta/\omega = 0/1$ and $\rho = -3$ in Chipman (2009, Fig.5).
Finally, as in Chipman (2009, f.3), we do not find that the parameters proposed in Mas-Colell et al. (1995, Exerc.17.D.1) and Hara et al. (1997, Exerc.17.D.1) (i.e., $\alpha/\beta = 2^{1/5}$, $\delta/\omega = 0$ and $\rho = -4$) result in the existence of an additional (asymmetric) competitive equilibrium. Condition (9) does not hold, although the necessary condition $\alpha/\beta > 1$ does. Indeed, the slope of the offer curves are positive at the symmetric equilibrium allocation defined by $R^* = 1$, but the slope of agent-A's offer curve is steeper than that of agent B. To obtain an example of multiple equilibrium that satisfies (9) one must take $\alpha/\beta > (-(1-\rho)/(1+\rho) = 12.86$, as numerically shown by Chipman.

[35] In the international trade literature, this entails that the specialization in the *domestic good* –i.e. whether $\omega >>> \delta > 0$ or the "full specialization" case $\delta = 0$– jointly with a preference bias towards the *domestic good*, might result in multiple equilibrium, relative price ratios. See specially the concluding remarks in Chipman (2009, Sec.5).

add some comments to this regard. Initially, these potential extensions require characterizing an agent's offer curve in the new settings as a simple regular open arc $C^i(\mathbf{R};\boldsymbol{\omega}^i) = d^i(\mathbf{R};\boldsymbol{\omega}^i)$ with $\mathbf{R} \in \mathcal{P} \equiv \mathcal{R}^{L-1}_{++}$, L is the number of commodities and $\mathbf{R}$ is the vector of relative prices. All notions defined in Section 3 and the straightforward properties of monotonicity and convexity apply to different axis directions. Although new shapes of offer curves can arise, *gross substitute* and $l-$*normal* offer curves can still be depicted, and those properties presented in section 4 would still satisfy. The most interesting results are those of uniqueness. Some of the findings in section 5 could be extended straightforwardly to the case of more agents and more commodities. For instance, if all agents' offer curve are simultaneously e_l-monotone for at least $L-1$ directions, then the equilibrium will be unique (as an extension of Theorem 5.1). This will be the case if all of the agents' offer curves satisfy the *gross substitute* property (Corollary 5.2.(i)), a result akin to that of Kehoe et al. (1991, Th.A). Two illustration of this result are the $L-$dimensional CES economies with $\rho \in [0,1]$, or the case of preferences represented by utility functions satisfying $\Psi^i_{ll'}(\mathbf{R};\boldsymbol{\omega}^i) \in [0,1]$ for all agents i and commodities l, $l' = 1,...L$. In addition, one may apply the same intuition in Propositions 5.5 and 5.12 to constrain the distribution of endowments for each shape of offer curves to guarantee a unique equilibrium. Finally, it is worth noting that restricting the shape of offer curves, by constraining the distribution of endowments and preferences, affects the profile of the *equilibrium manifold* (see Debreu 1970 or Balasko 1975). Whether the link between these two tools will provide more general findings on uniqueness of equilibrium is an open question.

Despite the interest of these extensions, the simplicity of our approach has a number of advantages. It provides useful geometric intuitions at the Edgeworth box, that would vanish as the number of commodities and agents enlarges. In addition, our simple framework is common enough in the literature of international trade (two-commodity two-country economies, with domestic and foreign commodities); the study of monetary or taxation issues (stationary two-agent two-period overlapping generations models, with old and young agents and with date-t and date-$t+1$ commodities); or computable equilibrium models (see Shoven and Whalley 1992). There, all our results have a straightforward application. Besides, analyzing groups of consumers could be promising, since two groups of agents are identified at each side of the market at each relative price: those that exhibit a positive net demand, and those that exhibit a negative net demand. The existence of two groups at each side of the market will not change after enlarging the number of individual types or commodities.

We conclude our work with an open problem: we were unable to find an example of *non-normal* offer curves in the literature, so the condition in Proposition 5.12 was left unchecked. Our work provides some clues on the type of preferences that characterize *non-normal* offer curves. Proposition 4.4 restricts the possibility of an offer curve being *non-normal* to the case in which the preference ordering cannot be a continuous representation of U that is homothetic or is strictly quasiconcave satisfying $U_{12}(\mathbf{x}) > 0$ for $R \in \mathcal{P}$ (so that $\Psi^i_l(R;\boldsymbol{\omega}^i) > 0$ for all agent i and commodity l). Also, Lemma 4.5 shows that any separable utility functions cannot shape *non-normal* offer curves. Additionally, Proposition A.1 in the Appendix shows that the pursued preferences must display every continuous representation satisfying $U_{12}(\mathbf{x}) < 0$. Whether or not the usual conditions on preferences restrict the existence of a utility function that allows for *non-normal* offer curves is still an open issue.

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Appendices

A.1 Notation

We will simplify notation by denoting $d^i \equiv d^i(R;\boldsymbol{\omega}^i)$ the agent i's optimal demand function for each $R \in \mathcal{P}$, and its derivatives, as $d_l \equiv d_l^i(R;\boldsymbol{\omega}^i)$ and $d_l' \equiv d_l^{i\prime}(R;\boldsymbol{\omega}^i)$ for $l = 1, 2$, with the gradient denoted as $\nabla d = (d_1', d_2')$; and, the first and second derivatives of the agent's utility function at the optimal allocation as $U_l \equiv U_l^i(d_1, d_2)$ and $U_{lk} \equiv U_{lk}^i(d_1, d_2)$ for $l,\ k = 1,\ 2$. The gradient evaluated at the optimal allocation is $\nabla U = (U_1, U_2)$, the Jacobian matrix is $D^2U = \begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$, and the Hessian matrix becomes

$$HU = \begin{pmatrix} U_{11} & U_{12} & U_1 \\ U_{21} & U_{22} & U_2 \\ U_1 & U_2 & 0 \end{pmatrix},$$

so that the bordered Hessian matrix is $det \mid HU \mid = U_1U_2[U_{12} + U_{21}] - U_1^2U_{22} - U_2^2U_{11}$.

To prove the results stated in this paper, it will be shown useful to differentiate two identities:

1. The *optimal condition*, $U_1 = RU_2$, whose differentiation is
$$U_{11}d_1' + U_{12}d_2' = U_2 + R[U_{21}d_1' + U_{22}d_2']. \tag{A.1}$$
2. The *budget line* at the optimal allocation, $R(d_1 - \omega_1^i) + (d_2 - \omega_2^i) = 0$, whose differentiation is $(d_1 - \omega_1^i) + Rd_1' + d_2' = 0$, so
$$Rd_1' + d_2' \quad = \quad \omega_1^i - d_1; \tag{A.2}$$
and whose second differentiation is
$$\frac{d_2''}{d_1''} \quad = \quad -R - 2\frac{d_1'}{d_1''}. \tag{A.3}$$

A.2 Properties of strictly-quasiconcave, twice-continuously differentiable utility functions

To prove some results, it will be useful to prove the following Proposition: any strictly quasiconcave, continuous rational ordering can be represented by a strictly concave continuous utility function.

Proposition A.1.
(i) Consider a continuous rational ordering represented by a strictly-quasiconcave, twice-continuously differentiable utility function $U : \mathcal{X} \to \mathcal{R}$. Then, there exists a bijective, continuous, strictly-monotone function $v : \mathcal{R} \to \mathcal{R}$ such that $V = v \circ U$ is a strictly-concave, twice-continuously differentiable utility function.
(ii) Consider a continuous rational ordering represented by a strictly quasiconcave, twice continuously differentiable utility function $U : \mathcal{X} \to \mathcal{R}$, satisfying $\Psi_1(R;\boldsymbol{\omega})\Psi_2(R;\boldsymbol{\omega}) \geq 0$ for all $R \in \mathcal{P}$. Then, there exists a bijective, continuous, strictly-monotone function $w : \mathcal{R} \to \mathcal{R}$ such that $W = w \circ U$ is a strictly-concave, twice-continuously differentiable utility function satisfying $W_{12}(\mathbf{x}) \geq 0$. In addition, it is satisfied that $\Psi_1(R;\boldsymbol{\omega}),\ \Psi_2(R;\boldsymbol{\omega}) \geq 0$ for all $R \in \mathcal{P}$.

Proof of Proposition A.1. To prove Part (i) of the Proposition, it is required to characterize the curvature of the function v. Denote by $v' \equiv v'(U(\mathbf{x})) > 0$ and $v'' \equiv v''(U(\mathbf{x}))$. Note that $\nabla V(\mathbf{x}) = v'\nabla U(\mathbf{x})$, and

$$D^2V(\mathbf{x}) = \begin{pmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{pmatrix} = \begin{pmatrix} v''U_1'^2 + v'U_{11} & v''U_1U_2 + v'U_{12} \\ v''U_1U_2 + v'U_{12} & v''U_2'^2 + v'U_{22} \end{pmatrix}.$$

If V is strictly concave, then it is required that $V_{11},\ V_{22} < 0$, and $\det \mid D^2V(\mathbf{x}) \mid > 0$ (see Mass-Collel et al. 1995, Th.M.C.2 p.937). These three conditions are, respectively, the following:

$$\frac{v''}{v'} < -\frac{U_{11}}{U_1^2};\ \frac{v''}{v'} < -\frac{U_{22}}{U_2^2}; \qquad \text{and,} \quad \frac{v''}{v'} < \frac{\det \mid D^2U \mid}{\det \mid HU(\mathbf{x}) \mid}, \tag{A.4}$$

with det $\mid D^2U \mid$ displaying an undefined sign, and det $\mid HU(\mathbf{x}) \mid > 0$ because of the quasiconcavity of the utility function (see Mass-Collel et al. 1995, Th.M.C.3). Thus, choosing a strictly monotone function v satisfying $\frac{v''}{v'} = \min\{-\frac{U_{11}}{U_1^2}; -\frac{U_{22}}{U_2^2}; \frac{\det|D^2U|}{\det|HU(\mathbf{x})|}\}$, we find a strictly concave function $V = v \circ U$.

To prove Part (ii) of the Proposition, we start by considering a strictly concave utility function V (i.e., V_{11}, $V_{22} < 0$ and det $\mid D^2V \mid > 0$), as this function exists as proved in Part (i), which satisfies that $V_{12} < 0$ (otherwise, the Proposition is straightforwardly satisfied). In fact, it is easy to prove that the condition in the Proposition for a quasiconcave function $\Psi_1^U(R;\boldsymbol{\omega})\Psi_2^U(R;\boldsymbol{\omega}) > 0$ is the same as $\Psi_1^V(R;\boldsymbol{\omega})\Psi_2^V(R;\boldsymbol{\omega}) > 0$. We have to prove that if this condition holds, then $W_{12} = w''V_1V_2 + w'V_{12} \geq 0$.

We will follow along the same development in the proof of Part (i) by substituting w for v, W for V, and V for U. Note that w''/w' can be positive from (A.4), as every term at the right-hand side is strictly positive. In addition, because of the strictly quasiconcavity of the utility function, then the determinant of the bordered matrix must be strictly positive, det $\mid HV(\mathbf{x}) \mid > 0$; that is $-[V_{11}/V_1^2] - [V_{22}/V_2^2] > -2[V_{21}/(V_2V_1)]$. Thus, substituting by (A.4), there may exist a function w such that $-[V_{11}/V_1^2] - [V_{22}/V_2^2] > 2w''/w' \geq -2[V_{21}/(V_2V_1)]$. This entails that $W_{12} \geq 0$ provided

$$\frac{\det \mid D^2V \mid}{\det \mid HV(\mathbf{x}) \mid} = \frac{\frac{V_{11}}{V_1^2}\frac{V_{22}}{V_2^2} - \left(\frac{V_{21}}{V_1V_2}\right)^2}{-\frac{V_{11}}{V_1^2} - \frac{V_{22}}{V_2^2} + 2\frac{V_{12}}{V_1V_2}} \geq -\frac{V_{12}}{V_1V_2}.$$

Operating, we find a second degree inequality,

$$\left(\frac{V_{12}}{V_1V_2}\right)^2 - \left[\frac{V_{22}}{V_2^2} + \frac{V_{11}}{V_1^2}\right]\left(\frac{V_{12}}{V_1V_2}\right) + \frac{V_{11}}{V_1^2}\frac{V_{22}}{V_2^2} \geq 0$$

The roots of the polynomial are V_{22}/V_2^2 and V_{11}/V_1^2, so the above inequality is satisfied if $\Psi_1^V(R;\boldsymbol{\omega})\Psi_2^V(R;\boldsymbol{\omega}) > 0$. Finally, because V_{11}, $V_{22} < 0$ and $V_{12} > 0$ then it is satisfied that $\Psi_1^V(R;\boldsymbol{\omega})$, $\Psi_2^V(R;\boldsymbol{\omega}) \geq 0$ and, consequently, $\Psi_1(R;\boldsymbol{\omega})$, $\Psi_2(R;\boldsymbol{\omega}) \geq 0$. This proves Part (ii). ∎

A.3 Proofs

Proof of Lemma 2.1. The proof is straightforward by substituting d_2' at (A.2) into (A.1). ∎

Proof of Theorem 3.2.
(i) Local monotonicity is a consequence of (A.2) evaluated at the autarkic allocation, so a local negative slope condition that guarantee monotonicity is satisfied.
(ii) First, we consider the case that the utility function is homothetic. Initially, we will prove that if preferences are homothetic any ray through origin only has one element at the offer curve. Assume that this is not the case. Then, there exists two relative prices $R_0 < R_1$ such that $\lambda d^i(R_0;\boldsymbol{\omega}^i) = d^i(R_1;\boldsymbol{\omega}^i)$ for $\lambda > 0$. Because of homotheticity, the marginal rate of substitution at both allocations coincide, $MRS(d^i(R_0;\boldsymbol{\omega}^i)) = MRS(d^i(R_1;\boldsymbol{\omega}^i))$. Because of optimality, the marginal rate of substitution at the optimal allocation equals the relative prices, $MRS(d^i(R_j;\boldsymbol{\omega}^i)) = R_j$ with $j = 0, 1$, so a contradiction arises. This entails that there exist no e_2-critical allocation, as $d_2'(R,\boldsymbol{\omega}^i) > 0$ for any $R \geq R_{\boldsymbol{\omega}^i}$, and no e_1-critical allocation, as $d_1'(R,\boldsymbol{\omega}^i) < 0$ for any $R \leq R_{\boldsymbol{\omega}^i}$. As a consequence, the arc segment $\overline{C}^i(R;\boldsymbol{\omega}^i)$ is e_2-monotone and the arc segment $\underline{C}(R;\boldsymbol{\omega}^i)$ is e_1-monotone.

Second, we consider the case that the utility function satisfies $\Psi_1^i(R;\boldsymbol{\omega}^i)\Psi_2^i(R;\boldsymbol{\omega}^i) \geq 0$ for all $R \in \mathcal{P}$, and then it is satisfied $\Psi_1^i(R;\boldsymbol{\omega}^i)$, $\Psi_2^i(R;\boldsymbol{\omega}^i) \geq 0$ for all $R \in \mathcal{P}$ by Proposition A.1.(ii). Because of the strictly quasiconcavy of preferences, it is satisfied $det \mid HU(d^i) \mid > 0$ (Mas-Colell et al. 1995, Theorem M.D.3). We can distinguish two cases. For any $R \in \underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$ it is satisfied that $\omega_2 - d_2 > 0$ and, because $\Psi_2^i(R;\boldsymbol{\omega}^i) > 0$, the slope of the demand function for commodity 1 at Lemma 2.1 is always negative, i.e. $d_1' < 0$ for $R \in \underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$; thus, $\underline{C}^i(R;\boldsymbol{\omega}^i)$ is e_1-monotone. Analogously, for any $R \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$, it is satisfied that $\omega_1 - d_1 > 0$ and, because $\Psi_1^i(R;\boldsymbol{\omega}^i) > 0$, the slope of the demand function for commodity 2 at Lemma 2.1 is always positive, i.e. $d_2' > 0$ for $R \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$; thus, $\overline{C}^i(R;\boldsymbol{\omega}^i)$ is e_2-monotone.
(iii) We can distinguish two cases. Recall from (ii) that if $\Psi_2^i(R;\boldsymbol{\omega}^i) \geq 0$, then $\underline{C}^i(R;\boldsymbol{\omega}^i)$ is e_1-monotone. Thus, it is only left to prove that if condition $\Psi_2^i(R;\boldsymbol{\omega}^i) \leq 1$ holds then $d_1' < 0$ for any $R \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$, so

that $\overline{C}^i(R;\boldsymbol{\omega}^i)$ is also e_1-monotone. The proof is straightforward as the slope of the demand function for commodity 1 at Lemma 2.1 is negative; that is,

$$1+\left[\frac{\omega_2^i}{d_2^i}-1\right]\Psi_2^i(R;\boldsymbol{\omega}^i)=\left[1-\Psi_2^i(R;\boldsymbol{\omega}^i)\right]+\frac{\omega_2^i}{d_2^i}\Psi_2^i(R;\boldsymbol{\omega}^i)>0. \tag{A.5}$$

Analogously, recall from (ii) that if $\Psi_1^i(R;\boldsymbol{\omega}^i)\geq 0$, then $\overline{C}^i(R;\boldsymbol{\omega}^i)$ is e_2-monotone. Thus, it is only left to prove that if condition $\Psi_1^i(R;\boldsymbol{\omega}^i)\leq 1$ holds then $d_2'>0$ for $R\in\underline{\mathcal{P}}_{\boldsymbol{\omega}^i}$; thus, $\underline{C}^i(R;\boldsymbol{\omega}^i)$ is also e_2-monotone. Again, the proof is straightforward as the slope of the demand function for commodity 2 at Lemma 2.1 is positive; that is

$$1+\left[\frac{\omega_1^i}{d_1^i}-1\right]\Psi_1^i(R;\boldsymbol{\omega}^i)=\left[1-\Psi_1^i(R;\boldsymbol{\omega}^i)\right]+\frac{\omega_1^i}{d_1^i}\Psi_1^i(R;\boldsymbol{\omega}^i)>0. \tag{A.6}$$

(iv) If the condition holds, the slope of the offer curve in (1) is negative for any $R\in\mathcal{P}$. This entails that there is no e_1- or e_2-critical allocations, so the offer curve is monotone.
This concludes the proof of Theorem 3.2. ∎

Proof of Proposition 3.3. Initially note that it may exist other e_l-critical allocations, as long as the arc segment is monotone, but only the extreme of the arc segment may be a *non-degenerate* critical one. We prove the results by contraction. To prove Part (i), let us assume that there exist an e_2-critical allocation at the arc segment defined by $[R_{\boldsymbol{\omega}^i},\widehat{R}]$, i.e. $d_2^{i\prime}=0$ at such an allocation. Since in this case $\omega_1^i-d_1^i>0$, a consequence of (A.2) is that $d_1^{i\prime}>0$ at this allocation, which is a contradiction with the fact that the arc segment is monotone. The proof of Part (ii) is analogous. Let us assume now that there exist an e_1-critical allocation at the arc segment defined by $[\widetilde{R},R_{\boldsymbol{\omega}^i}]$, i.e. $d_1^{i\prime}=0$ at such an allocation. Since $\omega_1^i-d_1^i<0$, a consequence of (A.2) is that $d_2^{i\prime}<0$ at this allocation. A contradiction arises. This concludes the proof of Proposition 3.3. ∎

Proof of Proposition 3.4. Consider that there exists a relative price $\breve{R}\in\mathcal{P}\backslash\{R_{\boldsymbol{\omega}^i}\}$ such that the allocation $d^i(\breve{R};\boldsymbol{\omega}^i)$ belonging to the offer curve is tangent to the budget line. Then, condition (A.2) requires that both individual demand functions exhibit an inflexion point at such an allocation, $d_1^{i\prime}(\breve{R};\boldsymbol{\omega}^i)=+\infty$ and $d_2^{i\prime}(\breve{R};\boldsymbol{\omega}^i)=+\infty$. The reason is the strict convexity of preferences. Dividing (A.1) by $d_1^{i\prime}$, it turns out that the determinant of the bordered Hessian is zero, i.e. $det\mid HU^i(d_1^i(\breve{R};\boldsymbol{\omega}^i))\mid=0$, which cannot be the case because of the strictly quasi-concavity of the utility function (Mas-Colell et al. 1995, Th.M.C.4). A contradiction that concludes the proof of Proposition 3.4. ∎

Proof of Proposition 3.5.
(i) Operating condition (2), we find

$$d_1^{i\prime}(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)d_1^{i\prime\prime}(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)\left[\frac{d_2^{i\prime\prime}(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)}{d_1^{i\prime\prime}(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)}-\frac{d_2^{i\prime}(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)}{d_1^{i\prime}(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)}\right]=-2\left(d_1^{i\prime}(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)\right)^2<0,$$

where the equality stems from conditions (A.2) and (A.3) both evaluated at the autarkic allocation.
(ii) From Proposition 3.3, denote $d^i(\widehat{R};\boldsymbol{\omega}^i)$ and $d^i(\widetilde{R};\boldsymbol{\omega}^i)$ as the only non-degenerate e_1- and e_2-critical allocations, respectively, at the monotone arc segment defined by $[\widetilde{R},\widehat{R}]$. We begin by proving that any offer curve $C^i(R;\boldsymbol{\omega}^i)$ is locally convex around the non-degenerate e_1-critical allocation $d^i(\widehat{R};\boldsymbol{\omega}^i)$. In this case $d_1^{i\prime}(\widehat{R};\boldsymbol{\omega}^i)=0$ and $\omega_1^i-d_1^i(\widehat{R},\boldsymbol{\omega}^i)>0$, so it is satisfied that $d_2^{i\prime}(\widehat{R};\boldsymbol{\omega}^i)>0$ by condition (A.2); then, the offer curve is locally convex if and only if $d_1^{i\prime\prime}(\widehat{R};\boldsymbol{\omega}^i)\geq 0$ by condition (2). Note that $d_1'(R)\leq 0$ for $R\in(R_{\boldsymbol{\omega}^i},\widehat{R})$ due to the arc segment defined for this interval is monotone (Proposition 3.3.(i)). By the Morse Lemma, the non-degenerate critical points belonging to a smooth function –such as the monotone arc segment of the offer curve– are isolated (see, e.g., Zomorodian 2005, Chap.5.3). Then, there exists an $\varepsilon>0$ satisfying that $d_1'(R;\boldsymbol{\omega}^i)<0$ and $d_1''(R;\boldsymbol{\omega}^i)>0$ at the left neighborhood $(\widehat{R}-\varepsilon,\widehat{R})$. This also entails that at the non-degenerate critical allocation, it is satisfied $d_1^{i\prime\prime}(\widehat{R};\boldsymbol{\omega}^i)\geq 0$. In addition, since we are assuming that $\widehat{R}$ defines a non-degenerate critical allocation (Definition 3), the inequality is strict. This

concludes the proof.

Analogously, we will prove that any offer curve $C^i(R;\boldsymbol{\omega}^i)$ is locally convex around the non-degenerate e_2-critical allocation $d^i(\widetilde{R};\boldsymbol{\omega}^i)$. In this case $d_2^{i\prime}(\widetilde{R};\boldsymbol{\omega}^i)=0$ and $\omega_1^i-d_1^i(\widetilde{R},\boldsymbol{\omega}^i)<0$, so it is satisfied that $d_1^{i\prime}(\widetilde{R};\boldsymbol{\omega}^i)<0$ by condition (A.2); then, the offer curve is locally convex if and only if $d_2^{i\prime\prime}(\widehat{R};\boldsymbol{\omega}^i)\geq 0$ by condition (2). Note that $d_2'(R)\geq 0$ for $R\in(\widetilde{R},R_{\boldsymbol{\omega}^i})$ due to the arc segment defined for this interval is monotone (Proposition 3.3.(ii)). Again, as the non-degenerate critical points are isolated by Morse Lemma, there exists an $\varepsilon>0$ satisfying that $d_2'(R;\boldsymbol{\omega}^i)>0$ and $d_1''(R;\boldsymbol{\omega}^i)>0$ at the right neighborhood $(\widetilde{R},\widetilde{R}+\varepsilon)$. This entails that at the non-degenerate critical allocation, it is satisfied $d_1^{i\prime\prime}(\widehat{R};\boldsymbol{\omega}^i)\geq 0$. In addition, since we are assuming that $\widetilde{R}$ defines a non-degenerate critical allocation (Definition 3), the inequality is strict. This concludes the proof, and the proof of Proposition 3.5. ∎

Proof of Proposition 3.7. Suppose it is not true, so the offer curve is concave but it is not monotone. From Definition 4, there must exist at least a non-degenerate e_l-critical allocation, one of those characterized by Proposition 3.3: whether the allocation $d^i(\widehat{R};\boldsymbol{\omega}^i)$ with $\widehat{R}>R_{\boldsymbol{\omega}^i}$ or $d^i(\widetilde{R};\boldsymbol{\omega}^i)$ with $\widetilde{R}<R_{\boldsymbol{\omega}^i}$. We prove that if the offer curve is concave there cannot exist any of these non-degenerate critical allocations. We prove the case that there exists an e_1-critical allocation, and the other proof is analogous.

If there exist a non-degenerate e_1-critical allocation and the offer curve is convex, then as $R\to+\infty$ the offer curve converges to an allocation $\mathbf{y}^*=(\omega_1^i,y_2^*)$. Note that $U(\mathbf{y}^*)>U(d^i(R;\boldsymbol{\omega}^i))$ for all $R\in\mathcal{P}$. Take now an allocation $\mathbf{y}^{**}=(\omega_1^i,y_2^*+\varepsilon)$ for any $\varepsilon>0$, and take a sequence $\{\mathbf{x}_n\}_{n=1}^{\infty}$ beginning at $x_1=d^i(\widehat{R};\boldsymbol{\omega}^i)$ that converges to $\mathbf{y}^{**}$. Observe that each allocation at the sequence can be associated to a budget line for some price ratio R, so that $R[\mathbf{x}_{n(R)}-\boldsymbol{\omega}^i]=0$ and –by optimality– $U(d^i(R;\boldsymbol{\omega}^i))>U(\mathbf{x}_{n(R)})$. On the other hand, by the strictly monotonicity of preferences $U(\mathbf{y}^{**})>U(\mathbf{y}^*)$. Then there must exist elements of the sequence at the upper contour set of $\mathbf{y}^*$; that is, there must exist an $\hat{n}$ such that $U(\mathbf{x}_n)>U(\mathbf{y}^*)$ for any $n\geq\hat{n}$. Choose any of this elements at the upper contour set, x_N with $N\geq\hat{n}$. Then, as before, we can associated a price ratio R', i.e. $N=n(R')$, such that $U(d^i(R';\boldsymbol{\omega}^i))>U(\mathbf{x}_{n(R')})$. But this is a contradiction by the transitivity of preferences, as the allocation $\mathbf{x}_{n(R')}$ is at the upper contour set of $\mathbf{y}^*$. This completes the proof of Proposition 3.7. ∎

Proof of Proposition 4.1. The proof of Parts (i)-(iii) are straightforward from previous results. Part (i) from the Definition 10 of normal offer curves with $[d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i)](\widehat{R}-R)<0$ [respectively, $[d_2^{i\prime}(R;\boldsymbol{\omega}^i)/d_1^{i\prime}(R;\boldsymbol{\omega}^i)](R-\widetilde{R})<0$], and from the Definition 4 of an e_l-monotone curve; Part (ii) from the Definitions 5, 6 and 10; and, Part (iii) form Proposition 3.5.(ii). Finally, the proof of Part (iv) is similar the one to Proposition 3.7, but by initially assuming that there is no inflexion point. ∎

Proof of Lemma 4.2. Proof of Part (i). As shown in (A.5), the sign of the slope of the demand function for commodity 1 at Lemma 2.1 depends on the sign of $1-\left[1-\frac{\omega_2^i}{d_2^i}\right]\Psi_2^i(R;\boldsymbol{\omega}^i)$. By Theorem 3.2.$(i)$ the term inside the brackets at the autarkic allocation is zero, so the slope is negative, i.e. $d_1'(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)<0$. As the relative price R tends to $+\infty$, commodity $l=2$ becomes relatively cheaper and by monotonicity of preferences its demand greatly increases, so the term inside the brackets converges to 1. Since by assumption the function $\Psi_2^i(R;\boldsymbol{\omega}^i)$ is non-decreasing and its value exceeds 1 for some R large enough, then eventually the slope of the demand function for commodity $l=2$ is positive, i.e. $d_1'(R;\boldsymbol{\omega}^i)>0$ for some R large enough. Thus, by continuity, there is a unique $\widehat{R}>R_{\boldsymbol{\omega}^i}$ such that the allocation is critical, $d_1^{i\prime}(\widehat{R};\boldsymbol{\omega}^i)=0$. Observe that there must exists a neighborhood of the critical allocation such that $[\widehat{R}-R]d_1'(R;\boldsymbol{\omega}^i)<0$ with $R\in\mathcal{B}(\widehat{R};\varepsilon)$ for some $\varepsilon>0$. In this neighborhood, the slope of the commodity 1 rises from a negative to a positive value, so d_1' is increasing and $d_1''(R;\boldsymbol{\omega}^i)>0$ for any $R\in\mathcal{B}(\widehat{R};\varepsilon)$. This entails that there is only one non-degenerate e_1-critical allocation for $\widehat{R}\in\overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$.

The proof of Part (ii) is analogous. As shown in (A.6), the sign of the slope of the demand function for commodity 2 at Lemma 2.1 depends on the sign of $1-\left[1-\frac{\omega_1^i}{d_1^i}\right]\Psi_1^i(R;\boldsymbol{\omega}^i)$. By Theorem 3.2.$(i)$ the term inside the brackets at the autarkic allocation is zero, so the slope is positive, i.e. $d_2'(R_{\boldsymbol{\omega}^i};\boldsymbol{\omega}^i)>0$. As the relative price R tends to 0 commodity $l=1$ becomes relatively cheaper and by monotonicity of preferences its demand greatly increases, so the term inside the brackets converges to 1. Since by assumption the function $\Psi_1^i(R;\boldsymbol{\omega}^i)$ is non-decreasing and its value exceeds 1 for some R large enough, then eventually the slope of the demand function for commodity $l = 1$ is negative, i.e. $d_2'(R;\boldsymbol{\omega}^i) > 0$ for some R small enough. Thus, by continuity, there is a unique $\widetilde{R} < R_{\boldsymbol{\omega}^i}$ such that the allocation is critical, $d_2^{i\prime}(\widetilde{R};\boldsymbol{\omega}^i) = 0$. Observe that there must exists a neighborhood of the critical allocation such that $[R - \widetilde{R}]d_2'(R;\boldsymbol{\omega}^i) < 0$ with $R \in \mathcal{B}(\widetilde{R};\varepsilon)$ for some $\varepsilon > 0$. In this neighborhood, the slope of the commodity 2 decreases from a positive to a negative value, so that d_2' is decreasing and $d_2''(R;\boldsymbol{\omega}^i) < 0$ for any $R \in \mathcal{B}(\widetilde{R};\varepsilon)$. This entails that there is only one non-degenerate e_2-critical allocation for $\widehat{R} \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^i}$. This completes the Proof of Lemma 4.2. ∎

Proof of Proposition 4.4. Part (i) and Part $(ii).(a)$ is a straightforward result from Theorem 3.2.(ii). Note that Part $(ii).(b)$ is also a straightforward result from the proof of Theorem 3.2.(ii) as, if preferences are homothetic any ray through origin only intersects once with the offer curve, and then Definition 8 is satisfied –i.e., there is at least one hyperplane through any $n-1$ points of the curve that contains no other points of the curve. ∎

Proof of Proposition 5.1. Consider the case that the agents' offer curve are e_1-monotone (the case e_2-monotone is analogous). Let $R^* \in [R_{\boldsymbol{\omega}^A}, R_{\boldsymbol{\omega}^B}]$ be an equilibrium relative price, and let $d^A(R^*;\boldsymbol{\omega}^A)$ and $d^B(R^*;\boldsymbol{\omega}^B)$ be the equilibrium allocations. Due to the agent A's and B's offer curve are e_1-monotone, then the lower that relative price the higher the quantity demanded of commodity 1 and the higher that relative price the lower the quantity demanded of commodity 1 (i.e. $d_1^i(R;\boldsymbol{\omega}^i) > d_1^i(R^*;\boldsymbol{\omega}^i)$ for any $R < R^*$, and $d_1^i(R;\boldsymbol{\omega}^i) < d_1^i(R^*;\boldsymbol{\omega}^i)$ for any $R > R^*$ and $i = A,\ B$). In the case that there exist another equilibrium for a lower relative price $R^{**} < R^*$ –thus, satisfying $d_1^A(R^{**};\boldsymbol{\omega}^A) > d_1^A(R^*;\boldsymbol{\omega}^A)$ and $d_1^B(R^{**};\boldsymbol{\omega}^B) > d_1^B(R^*;\boldsymbol{\omega}^B)$–, then the new equilibrium will not be feasible as $d_1^A(R^{**};\boldsymbol{\omega}^A)+d_1^B(R^{**};\boldsymbol{\omega}^B) > \omega_1$. Alternatively, in the case that there exist another equilibrium for a higher relative price $R^{**} > R^*$ –thus satisfying $d_1^A(R^{**};\boldsymbol{\omega}^A) < d_1^A(R^*;\boldsymbol{\omega}^A)$ and $d_1^B(R^{**};\boldsymbol{\omega}^B) < d_1^B(R^*;\boldsymbol{\omega}^B)$–, then the new equilibrium –satisfying $d_1^A(R^{**};\boldsymbol{\omega}^A) + d_1^B(R^{**};\boldsymbol{\omega}^B) < \omega_1$– cannot be optimal because of monotonicity of preferences. This completes the Proof of Proposition 5.1. ∎

Proof of Proposition 5.3. The proof is simple, after taking into account that each offer curve is monotone in at least one direction.
(i) If agent A's offer curve satisfies the gross substitute property, then the slope of the offer curve is negative –i.e., $d_2^{A\prime}(R;\boldsymbol{\omega}^A)/d_1^{A\prime}(R;\boldsymbol{\omega}^A) < 0$ for every $R \in \widehat{\mathcal{P}}$–, and the slope of the agent B's offer curve –whether negative or positive– is always greater than the slope of the agent A's offer curve, both evaluated at the competitive equilibrium with the equilibrium relative price $R^* \in \widehat{\mathcal{P}}$, so the term in the bracket is always negative.
(ii) If agent B's offer curve satisfies the gross substitute property, and the slope of the agent A's offer curve is positive (a negative slope was studied in (i)) –i.e., $d_2^{A\prime}(R;\boldsymbol{\omega}^A)/d_1^{A\prime}(R;\boldsymbol{\omega}^A) > 0$ for every $R \in \widehat{\mathcal{P}}$–, then the term in the bracket is always positive as the (positive) slope of the agent A's offer curve is always greater than the (negative) slope of the agent B's offer curve, both evaluated at the competitive equilibrium with the equilibrium relative price $R^* \in \widehat{\mathcal{P}}$.
(iii) Finally, we consider the case that the slope of agent A's and B's offer curve are both positive at the competitive equilibrium allocation, so agent A's and B's offer curve are $l = 2-$ and $l = 1-$normal, respectively. In this case, the equilibrium allocation is unique if the term in the bracket is positive; that is, the (positive) slope of the agent A's offer curve is greater than the (positive) slope of the agent B's offer curve, both evaluated at the competitive equilibrium with the equilibrium relative price $R^* \in \widehat{\mathcal{P}}$. Otherwise, by the monotonicity of both offer curves, both offer curves will eventually intersect each other twice. This concludes the proof of Proposition 5.3. ∎

Proof of Proposition 5.5. Since $\mathcal{C}^A(R;\boldsymbol{\omega}^A)$ is e_2-monotone, then the arc segment $\widehat{\mathcal{C}}^A(R;\boldsymbol{\omega}^A)$, with the non-degenerate e_1-critical allocation defined by $\widehat{R}^A \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^A}$ as endpoint, is also is also e_2-monotone. If conditions (4) holds, then there exists an arc segment of $\widehat{\mathcal{C}}^B(R;\boldsymbol{\omega}^B)$, with endpoints at the autarkic allocation and at the non-degenerate critical allocation, which is e_2-monotone. Thus, two e_l-monotone curves with opposite slope sign only intersect once (Theorem 5.1). Analogously, since $\mathcal{C}^B(R;\boldsymbol{\omega}^B)$ is an e_1-monotone curve, and if (5) holds then there exists an arc segment of $\widehat{\mathcal{C}}^B(R;\boldsymbol{\omega}^B)$, with endpoints at the autarkic allocation and at the non-degenerate e_2-critical allocation defined by $\widetilde{R}^B \in \overline{\mathcal{P}}_{\boldsymbol{\omega}^A}$, which is also

e_1-monotone. Again, two e_l-monotone curves with opposite slope sign only intersect once (Theorem 5.1). This concludes the proof of Proposition 5.5. ∎

Proof of Corollary 5.10.(c) **and** (d)**.**

Proof of Corollary 5.10.(c) To prove Corollary 5.10.(c) recall that we are assuming that (i) Assumption 1 holds; (ii) that preferences are additively separable, i.e. $U^i(\mathbf{x}^i) = \lambda_1 u(x_1^i) + \lambda_2 u(x_2^i)$ with $\lambda_1, \lambda_2 > 0$, so the absolute risk aversion becomes $\Psi_l^i(R;\boldsymbol{\omega}^i)/d_l^i(R;\boldsymbol{\omega}^i) = -u''(d_l^i(R;\boldsymbol{\omega}^i))/u'(d_l^i(R;\boldsymbol{\omega}^i))$ for commodities $l = 1$ and 2 –which take positive values because of the strictly convexity of u–; (iii) the utility function satisfies non-decreasing absolute risk aversion for commodities $l = 1$ and 2, i.e. if $d_l^i(R'';\boldsymbol{\omega}^i) \geq d_l^i(R';\boldsymbol{\omega}^i)$ then $\Psi_l^i(R'';\boldsymbol{\omega}^i)/d_l^i(R'';\boldsymbol{\omega}^i) \leq \Psi_l^i(R';\boldsymbol{\omega}^i)/d_l^i(R';\boldsymbol{\omega}^i)$ for $l = 1, 2$; and, (iv) the endowment distribution satisfies $\omega_1^A \leq \boldsymbol{\omega}_1/2 \leq \omega_1^B$ and $\omega_2^A \geq \boldsymbol{\omega}_2/2 \geq \omega_2^B$.

Observe that by (i) and (iv), every competitive equilibrium satisfies $d_l^A(R;\boldsymbol{\omega}^i) < \boldsymbol{\omega}_1/2 < d_l^B(R;\boldsymbol{\omega}^i)$ and $d_2^A(R;\boldsymbol{\omega}^i) > \boldsymbol{\omega}_2/2 > d_2^B(R;\boldsymbol{\omega}^i)$. Accordingly, by (iii) every competitive equilibrium also satisfies $\frac{\Psi_1^A(R;\boldsymbol{\omega}^A)}{d_1^A(R;\boldsymbol{\omega}^A)} > \frac{\Psi_1^B(R;\boldsymbol{\omega}^B)}{d_1^B(R;\boldsymbol{\omega}^B)}$ and $\frac{\Psi_2^B(R;\boldsymbol{\omega}^B)}{d_2^B(R;\boldsymbol{\omega}^B)} > \frac{\Psi_2^A(R;\boldsymbol{\omega}^A)}{d_2^A(R;\boldsymbol{\omega}^A)}$, respectively. Taking into account by (ii) that absolute risk aversion function takes positive values, we can multiply both inequalities to find that every competitive equilibrium in an economy satisfying conditions (i)-(iv) must also satisfy the expression in Lemma 5.4, a sufficient condition to satisfy the necessary condition (3) in Proposition 5.3. This means that, if every equilibrium in the economy described in Corollary 5.10.(c) must satisfy (3), and (3) is a condition satisfied when an equilibrium allocation is unique, then the economy described in the corollary has only one competitive equilibrium. This concludes the proof of Corollary 5.10.(c).

Proof of Corollary 5.10.(d) We begin by substituting in the expression in Lemma 5.4 the constant values of the relative risk aversion and the market clearing condition $d_l^B = \boldsymbol{\omega}_l - d_l^A$ for $l = 1, 2$. After operating, we find that

$$d_2^A \geq \frac{\Psi_2^A \Psi_1^B}{\frac{\Psi_1^A \Psi_2^B}{d_1^A} + [\Psi_2^A \Psi_1^B - \Psi_1^A \Psi_2^B]} \frac{\boldsymbol{\omega}_2}{\boldsymbol{\omega}_1}.$$

If the condition at Corollary 5.10.(d) holds, then the previous inequality also holds as $d_2^A \leq \omega_1^A$ and $d_2^A \geq \omega_2^A$, , because of Assumption 1. Finally, since the condition in described in Corollary 5.10.(d) is satisfied at every allocation at the trading price set $\widehat{\mathcal{P}}$, it indeed is satisfied for every equilibrium allocation. Hence, as every equilibrium in the economy described in Corollary 5.10.(d) must satisfy (3), and (3) is a condition satisfied when an equilibrium allocation is unique, then the economy described in the corollary has only one competitive equilibrium. This concludes the proof of Corollary 5.10.(d). ∎

Proof of Proposition 6.1. We begin by showing that the utility function $U(x_1, x_2) = ax_1 - x_2^{-b}/b$ with $a, b > 0$ is quasi-concave. The gradient is $\nabla U(x_1, x_2) = (a, x_2^{-b-1})$, and the second derivative matrix is

$$D^2U(x_1, x_2) = \begin{pmatrix} 0 & 0 \\ 0 & -(b+1)x_2^{-b-2} \end{pmatrix}.$$

The utility function is quasi-concave as it satisfies that the determinant of the bordered Hessian is positive, i.e. $\det HU(x_1, x_2) = a^2(b+1)x_2^{-(b+2)} > 0$.

To prove Part (i), that for these preferences and any endowment the offer curve is $l = 2-$*normal*, it is enough to show that the arc segment $\underline{C}(R;\boldsymbol{\omega})$ satisfies the *gross substitute* property, while its complement arc segments $\overline{C}(R;\boldsymbol{\omega})$ is *normal*. Initially, since $\Psi_1^i(R;\boldsymbol{\omega}^i) = 0$ and $\Psi_2^i(R;\boldsymbol{\omega}^i) = 1 + b > 1$ –i.e., constant and non-decreasing– for all $R \in \mathcal{P}$, then $\underline{C}(R;\boldsymbol{\omega})$ satisfies the *gross substitute* property (Definition 5) as the slope of the offer curve (1) is always negative for $\omega_2 > d_2$. In addition, Lemma 4.2.(i) is satisfied and, accordingly, $\overline{C}(R;\boldsymbol{\omega})$ is $l = 2-$*normal*. Lemma 4.2.(i) also entails that there exists a unique non-degenerate e_1-critical allocation satisfying $d_1'(\widehat{R}) = 0$, i.e. a root of $[1 - \Psi_2^i + (\omega_2/d_2(\widehat{R}))\Psi_2^i] = 0$, with the individual demand function being $d(R) = \big(\omega_1 - [(R/a)^{1/(b+1)} - \omega_2]/R; (R/a)^{1/(b+1)}\big)$. This proves Part (ii). To prove Part (iii), the inflection point is the root of (2).

The proof is analogous for the utility function $U(x_1, x_2) = -x_1^{-b}/b + ax_2$ with $a, b > 0$. Here $\Psi_1^i(R;\boldsymbol{\omega}^i) = 1 + b > 1$ and $\Psi_2^i(R;\boldsymbol{\omega}^i) = 0$ for all $R \in \mathcal{P}$, $\overline{C}(R;\boldsymbol{\omega})$ satisfies the *gross substitute* property (Definition 5) as the slope of the offer curve (1) is always negative for $\omega_1 < d_1$, Lemma 4.2.(ii) is satisfied so $\underline{C}(R;\boldsymbol{\omega})$ is $l = 1-$*normal*, and $d(R) = \big((Ra)^{-1/(b+1)}; \omega_2 - [(Ra)^{-1/(b+1)} - \omega_1]R\big)$. ∎

Proof of Proposition 6.5. The demand functions for agents A and B are defined in the proof of Proposition 6.1. The agent's gradient of the demand function are $\nabla d^{A\prime} = ([\,[b/(b+1)](R/a)^{1/(b+1)} - \omega_2^A\,]/R^2;\ [1/[(b+1)](R^b a)^{-1/(b+1)})$ and $\nabla d^{B\prime} = (-[1/(b+1)](R^{b+2}a)^{-1/(b+1)};\ [b/(b+1)](R^{2b+1}a)^{-1/(b+1)} + \omega_1^B)$. The slope of the agents' offer curve evaluated at the equilibrium price $R^* = 1$ for the supersymmetric case is:

$$\frac{d_2^{A\prime}(1)}{d_1^{A\prime}(1)} = \left(b\left[1 - (\widehat{R}^A)^{\frac{1}{b+1}}\right]\right)^{-1}; \quad \text{and,} \quad \frac{d_2^{B\prime}(1)}{d_1^{B\prime}(1)} = b\left[1 - (\widetilde{R}^B)^{-\frac{1}{b+1}}\right]. \tag{A.7}$$

Proposition 6.5 holds, provided conditions in Proposition 5.11 are satisfied; that is, provided (a) both slopes in (A.7) are positive at $R^* = 1$, a condition that is satisfied because $\widehat{R}^A < 1$ and $\widetilde{R}^B > 1$; and, (b) $d_2^{A\prime}(1)/d_1^{A\prime}(1) < d_2^{B\prime}(1)/d_1^{B\prime}(1)$. Observe that, since $\widehat{R}^A = (\widetilde{R}^B)^{-1}$, a condition guaranteeing a higher slope in (b) is $\left(b[1 - [\widehat{R}^A]^{1/(b+1)}]\right)^2 - 1 > 0$, that is, whenever $\widehat{R}^A < (1 - 1/b)^{b+1}$ or $\widehat{R}^A > (1 + 1/b)^{b+1}$. Combining (a) –i.e., $\widehat{R}^A < 1$– and (b), only the former condition holds. Thus condition $\widehat{R}^A > (1 + 1/b)^{b+1}$ is the one in the Proposition to be satisfied. ∎

Proof of Proposition 6.6. We begin by showing that the CES utility function is quasi-concave. The gradient is $\nabla U(x_1, x_2) = (a x_1^{\rho-1}, b x_2^{\rho-1})$, and the second derivative matrix is

$$D^2 U(x_1, x_2) = \begin{pmatrix} a(\rho-1)x_1^{\rho-2} & 0 \\ 0 & b(\rho-1)x_2^{\rho-2} \end{pmatrix}.$$

The utility function is quasi-concave as it satisfies that the determinant of the bordered Hessian is positive, i.e. $\det HU(x_1, x_2) = (1-\rho)ab x_1^{\rho-2} x_2^{\rho-2}[a x_1^\rho + b x_2^\rho] > 0$.
Part (i) is a straightforward consequence of Lemma 4.5 and Theorem 3.2.(iii), since $\Psi_1^i(R; \boldsymbol{\omega}^i) = \Psi_2^i(R; \boldsymbol{\omega}^i) = 1 - \rho \leq 1$ provided $\rho \geq 0$, for all $R \in \mathcal{P}$.
To prove Part $(ii.1)$ –i.e., if $\rho < 0$, then the offer curve is normal–, it is enough to show that the two complement arc segments $\overline{C}(R; \boldsymbol{\omega})$ and $\underline{C}(R; \boldsymbol{\omega})$ are both normal. If $\rho < 0$ then $\Psi_1^i(R; \boldsymbol{\omega}^i) = \Psi_2^i(R; \boldsymbol{\omega}^i) = 1 - \rho > 1$ –i.e., constant and non-decreasing–, and from Lemma 4.2 both $\overline{C}(R; \boldsymbol{\omega})$ and $\underline{C}(R; \boldsymbol{\omega})$ are both normal.

Finally, Lemma 4.2 also entails that there exists a unique non-degenerate e_1−critical allocation satisfying $d_1'(\widehat{R})$, i.e. a root of $\phi_1(R) = [1 - \Psi_2^i + (\omega_2/d_2(R))\Psi_2^i] = 0$, and there exists a unique non-degenerate e_1−critical allocation satisfying $d_2'(\widehat{R}) = 0$, i.e. a root of $\phi_2(R) = [1 - \Psi_1^i + (\omega_1/d_1(R))\Psi_1^i]$, where the individual demand function is $(d_1(R), d_2(R)) = ((Rb/a)^{1/(\rho-1)}[R\omega_1 + \omega_2]/[(R^\rho b/a)^{1/(\rho-1)} + 1]; [R\omega_1 + \omega_2]/[(R^\rho b/a)^{1/(\rho-1)} + 1])$. There is no simple analytical expression for both critical allocations, but considering $\phi_1(\widehat{R}) = 0$ at the demand function, it is easy to find that $d_2(\widehat{R}) = [(1-\rho)/(-\rho)]\omega_2^A$; and, analogously, considering $\phi_1(\widetilde{R}) = 0$ at the demand function, we can find that $d_1(\widetilde{R}) = [(1-\rho)/(-\rho)]\omega_1^A$.

This proves Part $(ii.2)$, and concludes the proof of Proposition 6.6. ∎

Proof of Proposition 6.9. The demand functions for agents A and B are defined in the proof of Proposition 6.6. Computing the agent i's gradient of the demand function, and thus the slope of the agents' offer curve at the equilibrium price $R^* = 1$ for the supersymmetric case, we find:

$$d_1^{i\prime}(1) = d_1^i(1)\frac{\left[\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}} + 1\right]\boldsymbol{\omega}_1^i - \left[\frac{1}{1-\rho^i} + \left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}}\right][\boldsymbol{\omega}_1^i + \boldsymbol{\omega}_2^i]}{\left[\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}} + 1\right][\boldsymbol{\omega}_1^i + \boldsymbol{\omega}_2^i]} \tag{A.8}$$

$$d_2^{i\prime}(1) = d_2^i(1)\frac{\left[\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}} + 1\right]\boldsymbol{\omega}_1^i + \frac{\rho^i}{1-\rho^i}\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}}[\boldsymbol{\omega}_1^i + \boldsymbol{\omega}_2^i]}{\left[\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}} + 1\right][\boldsymbol{\omega}_1^i + \boldsymbol{\omega}_2^i]} \tag{A.9}$$

$$\frac{d_2^{i\prime}(1)}{d_1^{i\prime}(1)} = \frac{\left[\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}} + 1\right]\boldsymbol{\omega}_1^i + \frac{\rho^i}{1-\rho^i}\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}}[\boldsymbol{\omega}_1^i + \boldsymbol{\omega}_2^i]}{\left[\left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}} + 1\right]\boldsymbol{\omega}_1^i - \left[\frac{1}{1-\rho^i} + \left(\frac{b^i}{a^i}\right)^{\frac{1}{\rho^i-1}}\right][\boldsymbol{\omega}_1^i + \boldsymbol{\omega}_2^i]}\left(\frac{b^i}{a^i}\right)^{-\frac{1}{\rho^i-1}}. \tag{A.10}$$

Consider the parameters for the supersymmetric economy (see Corollary 6.8): $\boldsymbol{\omega}^A = (\omega, \delta)$ and $\boldsymbol{\omega}^B = (\delta, \omega)$ with $\omega \geq \delta$, $a^A = b^B = \alpha^{1-\rho}$ and $b^A = a^B = \beta^{1-\rho}$ with $\alpha \geq \beta$, and $\rho^A = \rho^B = \rho$. Recall that Assumption 1 states that $\alpha\delta - \beta\omega > 0$.

The results holds provided it is satisfied simultaneously that (a) derivatives (A.8)-(A.9) are both positive for $i = A$ –i.e., $d_1^{A\prime}(1)$, $d_2^{A\prime}(1) > 0$– and are both negative for $i = B$ –i.e., $d_1^{B\prime}(1)$, $d_2^{B\prime}(1) < 0$; and, (b) $d_2^{A\prime}(1)/d_1^{A\prime}(1) < d_2^{B\prime}(1)/d_1^{B\prime}(1)$. Concerning (a), observe first that Proposition 6.6 states that if $\rho < 0$, then the two complement arc segments $\underline{C}(R; \boldsymbol{\omega})$ and $\overline{C}(R; \boldsymbol{\omega})$ are $l = 1-$ and $l = 2-$normal respectively. This means that $d_2^{A\prime}(R) > 0$ for all $R \geq R_{\boldsymbol{\omega}^A}$ and $d_1^{B\prime}(R) < 0$ for all $R \leq R_{\boldsymbol{\omega}^B}$, and indeed it is the case for $R = 1$. Interestingly, the same condition

$$1 - \frac{\omega/(\omega+\delta)}{(\alpha/\beta)/[1+(\alpha/\beta)]} < \frac{1}{1-\rho} \tag{A.11}$$

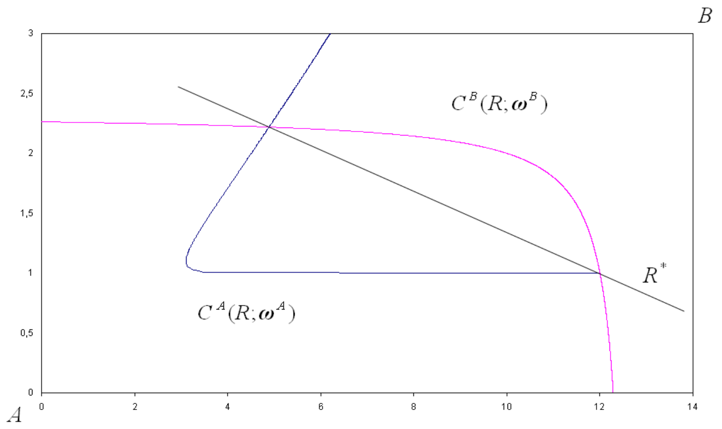
is found from (A.8)-(A.9) for $d_2^{A\prime}(1) > 0$ and $d_1^{B\prime}(1) < 0$. Observe, however, that the left hand-side term is always negative because of Assumption 1, so that (A.11) always holds. It is left to prove that the numerator of (A.8) is positive for $i = A$ and the numerator of (A.9) is negative for $i = B$; that is, respectively, to prove that $d_1^{A\prime}(1) > 0$ and $d_2^{B\prime}(1) < 0$ are satisfied. Again, from (A.8)-(A.9), the same condition is found for both:

$$\frac{1}{1-\rho} < 1 - \frac{\alpha}{\beta}\frac{\delta/(\omega+\delta)}{(\alpha/\beta)/[1+(\alpha/\beta)]}. \tag{A.12}$$

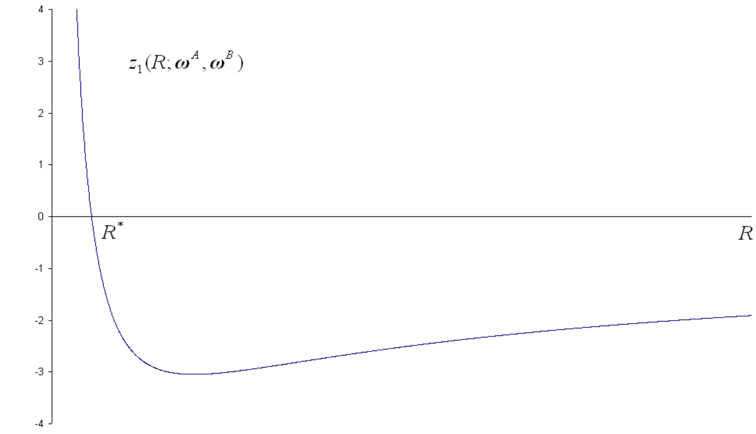
Because of Assumption 1, the condition at the right hand-side of (A.12) is always lower than the condition at the left hand-side of (A.11), so there always exists a region of parameter ρ such thus (a) is satisfied.

Concerning (b), after substituting the parameters in (A.10), $d_2^{A\prime}(1)/d_1^{A\prime}(1) < d_2^{B\prime}(1)/d_1^{B\prime}(1)$ holds provided the condition (9) at the Proposition is satisfied. Observe that the upper threshold of this condition falls right between the lower threshold in (A.11) and the upper threshold in (A.12).

Finally, observe that in the supersymmetric mirrow CES economy, if the competitive equilibrium defined by the relative ratio $R^* = 1$ is not in the conditions of Proposition 5.11 but in those of Proposition 5.3, so that such an equilibrium is unique. This proves Proposition 6.9. ∎

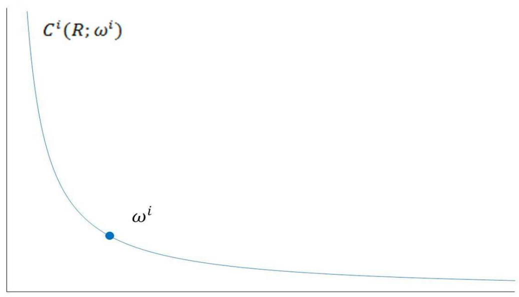


(a) **Edgeworth Box and a unique equilibrium.**

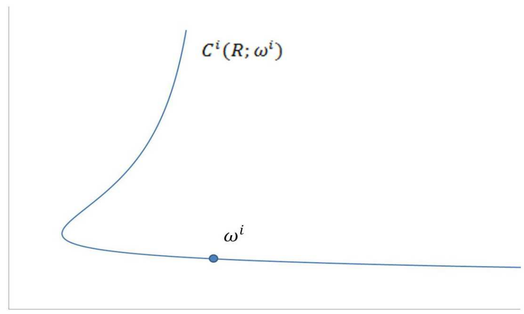


(b) **Non-monotone excess of demand function and a unique equilibrium.**

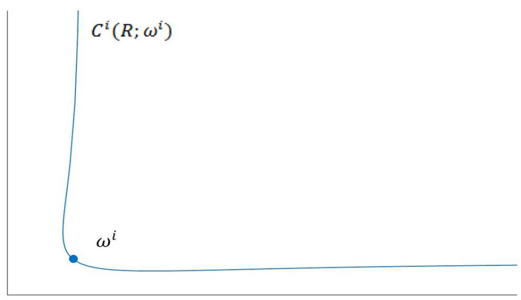
Figure 1: **An illustration of a unique equilibrium price ratio R^* with e_2-monotone offer curves and a non-monotone aggregate excess of demand function.** The simultaneous monotonicity in direction of vertical (e_2-)axis of individual offer curves –$\mathcal{C}^A(R;\boldsymbol{\omega}^A)$ and $\mathcal{C}^B(R;\boldsymbol{\omega}^B)$– results in a unique competitive equilibrium (Theorem 5.1), but it need not imply the monotonicity of the aggregate excess of demand ($z_1(R;\boldsymbol{\omega}^A,\boldsymbol{\omega}^B)$). (Illustration with agent A's CES preferences $U(\mathbf{x}^A) = 1024\,x_1^{-10} + x_2^{-10}$ and endowments $\boldsymbol{\omega}^A = (12,1)$; and, agent B's *Cobb-Douglas* preferences $U(\mathbf{x}^B) = x_1^{2/3}x_2^{1/3}$ and endowments $\boldsymbol{\omega}^B = (2,2)$.)



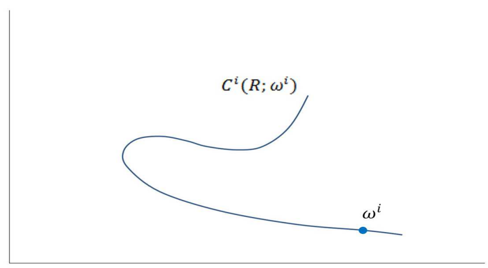
(a) **Shape 1. A gross substitute offer curve** (see Mas-Colell 1991, Fig.1(b); and Grandmond 1985, Fig.1.a). CES utility function: $U(x_1, x_2) = ax_1^\rho + bx_2^\rho$, with $a = 1$, $b = 1$, $\rho = 0.5$ and $\omega^i = (2, 2)$.



(b) **Shape 2. A normal offer curve with one critical allocation: An $l = 2-$normal offer curve** (see Marshall 1879, Fig.4; Mas-Colell 1991, Fig.1(a); Mas-Colell et al. 1995, Fig.15.B.5; and Grandmond 1985, Fig.1.b). Shapley-Shubik utility function: $U(x_1, x_2) = ax_1 - x_2^{-b}/b$, with $a = 0.1$, $b = 2$ and $\omega^i = (2, 1)$.

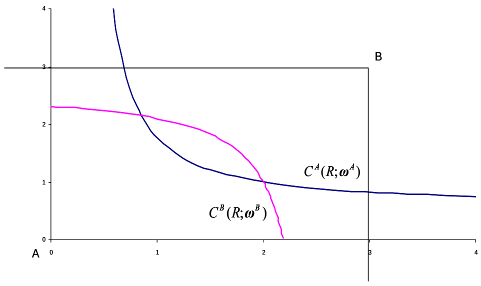


(c) **Shape 2. A normal offer curve with two critical allocations.** (see Marshall 1879, Fig.4; Mas-Colell 1991, Fig.1(a); Mas-Colell et al. 1995, Fig.15.B.5; and Grandmond 1985, Fig.1.b). CES utility function: $U(x_1, x_2) = ax_1^\rho + bx_2^\rho$, with $a = 1$, $b = 1$, $\rho = -4$ and $\omega^i = (2, 2)$.

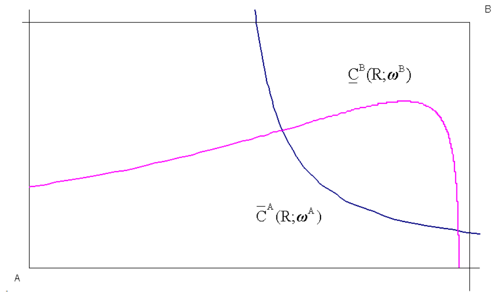


(d) **Shape 3. A non-normal offer curve** (See Johnson 1958, Fig.5; Mas-Colell 1991, Fig.1(c); and Mas-Colell et al. 1995, Fig.2.E.4.).

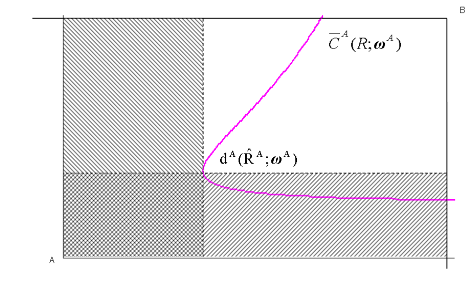
Figure 2: **Offer curve: some possible shapes.**



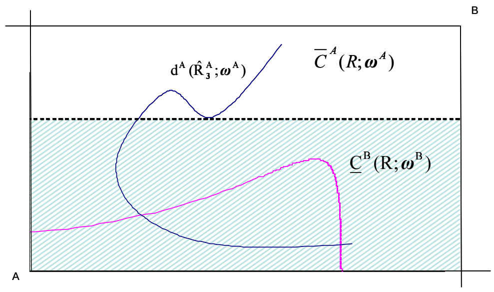
(a) **Gross substitute offer curves: Corollary 5.2.(i).**



(b) **Gross substitute and** l = 1−**normal offer curves: Corollary 5.2.(ii).**



(c) **Opposite, normal offer curves: Proposition 5.5.** Conditions (4) and (5) entail that $d^B(\widetilde{R}^B;\boldsymbol{\omega}^B)$ is located at the western and southern area of the non-degenerate critical allocation $d^A(\widehat{R}^A;\boldsymbol{\omega}^A)$, respectively.



(d) l = 1−**normal and non-normal offer curves: Proposition 5.12.** $d^B(\widetilde{R}^B;\boldsymbol{\omega}^B)$ is located at the southern area of the non-degenerate critical allocation $d^A(\widehat{R}^A;\boldsymbol{\omega}^A)$.

Figure 3: **Offer curves and uniqueness.**