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Abstract. This article surveys the conditions under which it is possible to represent a continuous preference ordering using utility functions. We start with a historical perspective on the notions of utility and preferences, continue by defining the mathematical concepts employed in this literature, and then list several key contributions to the topic of representability. These contributions concern both, the preference orderings and the spaces where they are defined. For any continuous preference ordering, we show the need for separability and the sufficiency of connectedness and separability, or second-countability, of the space where it is defined. We emphasize the need of separability by showing that in any non-separable metric space there are continuous preference orderings without utility representation. However, by reinforcing connectedness, we show that countably boundedness of the preference ordering is a necessary and sufficient condition for the existence of a (continuous) utility representation. Lastly, we discuss the special case of strictly monotonic preferences.

Keywords. Preference ordering; Preference representation; Utility function

1. Introduction

The hypothesis that rational behavior can be represented as the maximization of a utility function is central to economic science. This article is about the conditions required for the existence of such representation. We aim to provide a summary of the main results on this topic which, despite its longevity, remains an area of active research. The introduction, even though this is a piece of mathematical economics, is devoted to offering a historical perspective on the notions of utility and preference.

The concept of *utility* commonly refers to the satisfaction received, or pain avoided, from consuming a good or service. Its usage can be at least traced back to Aristotle (Kauder, 1953; Gordon, 1964), and it was already employed extensively by Scholastics and Mercantilists, although it was not until the twentieth century that it acquired the meaning it has today. Up until the late nineteenth century economists understood utility as a measure of individuals' welfare, an attribute to rate human happiness. It was thought of as a quantifiable psychic magnitude, and a scale was even created for that matter, with "utils" being the name given to its units. Yet, how utility could actually be measured was never satisfactorily defined, even thoguh much attention was devoted to the task (most notably from the Marginalist school; see Stigler, 1950).

Towards the end of the Victorian era this understanding of utility started to be seen as vague and subjective, a notion not suitable for the scientific discipline that the economy was wanted to be. The lack of a "scientific" method to measure utility motivated some economists to change the approach, the first of these being Irving Fisher (1892) and Vilfredo Pareto (1900, 1901, 1906, 1911) (see also Lewin, 1996). These authors realized that with regard to utility, focusing on the *order* in which options are ranked —instead of on how much they differ from each other— served very well for the treatment of the problems to which utility theory was conventionally applied, and in this way utility no longer had to be thought of as a psychological entity measurable in its own right (Strotz, 1953). This approach is known as *ordinalism*, and paved the way for modern microeconomics; in fact, the resolvability of the demand problem is behind the ordinalist approach, and so is the representability problem. Then, instead of assessing the unobservable pleasure obtained (or pain avoided) from consuming a commodity, the object of study moved to the observable choice behavior between commodities, viz., the individuals' *preferences* (Varian, 2010). With this new focus, economic theory would no longer fall within the reign of psychics, trying to measure a subjective and intangible notion, but belong to the study of concrete, observable phenomena, just as physics did with the movement of bodies (Pareto, 1906).

The attempt to bring economics into scientific grounds was most enforced by the critical introduction of mathematical discourse, which would shape the discipline thereafter. Nevertheless, even though that much importance was given to it from the outset, it took decades until mathematics was properly applied to utility theory. Back when the move towards ordinalism took place it was blithely assumed that a *preference ordering* could always be measured numerically. This is, that it would always be possible to assign a real number to every consumption bundle available to a consumer in *representation* of the order in which she prefers them. This assigning is precisely what a *utility function* does, and is very helpful in solving the demand problem. Sadly, as we now know, it is not true that it can always be used to represent a preference ordering.

Building upon the works of Pareto, Eugene Slutsky (1915), John Hicks and Roy Allen (1934) and Paul Samuelson (1938), among others, contributed to make ordinalism the dominant approach in consumer theory, and explored the possibilities opened up by the study of preferences. Still, it was not until the work of Wold (1943) that due attention was paid to the conditions under which a utility function could represent some ranking of a consumer's preferences. The work of Wold was followed by that of Debreu and many more authors after him who, over the following decades and up to the present time, have been trying to refine and generalize the results on the matter. Reviewing this literature, regarding the conditions that guarantee a continuous utility representation of a preference ordering, is the object of this survey.

The paper is organized as follows: in Section 2 we present the technical definitions of the concepts employed. Section 3 starts with a basic example of how to represent a preference ordering, and continues with the typical counterexample of a non-representable preference. We use these cases to emphasize, in Subsection 3.3, the relevance of the assumptions of connectedness and separability (or second-countability) required by the "classical" existence result on preference representation. In Section 3.4 we point out that in any non-separable metric space there are continuous, transitive and complete preference orderings without utility representation. This could be problematic, as in many economic applications the consumption sets may be non-separable. This is the case, for instance, in the analysis of resource allocation over time or states of nature (Bewley, 1972) or in models exhibiting commodity differentiation (Mas-Colell, 1975). Nonetheless, in Subsection 3.5 we show that it is possible to overcome this problem by reinforcing connectedness of the space of alternatives and adding an assumption of boundedness of the continuous preference. In Section 4 we consider strictly monotonic preferences emphasizing the scenario where the set of commodities is uncountable. It is shown that strictly monotonic preference orderings always exist, and that if the commodity space is "rich enough" they cannot be continuous nor representable by any utility function. We conclude in Section 5 with a summary of the main results.

2. Definitions

A preference ordering, say $\preccurlyeq$, on a set X, is a *transitive* and *complete* (and thus, *reflexive*) binary relation on X. Hence, for every $x, y \in X$, either $x \preccurlyeq y$ or $y \preccurlyeq x$. Moreover, if $x \preccurlyeq y \preccurlyeq z$, then $x \preccurlyeq z$. We read $x \preccurlyeq y$ as "y is at least as preferred as x". In much of microeconomic theory individual preferences are assumed to be *rational*, with completeness and transitivity being necessary and sufficient conditions to guarantee this. However, weaker forms of rationality are possible, such as *quasitransitive* and *acyclic* preferences. Yet, since this paper is about preferences that can be represented by utility functions, which must be both transitive and complete, no weaker forms of rationality will be discussed.[1]

Two important relations derived from $\preccurlyeq$ are $x \prec y$, which reads "y is strictly preferred to x" and it is defined by $x \prec y$ if and only if $x \preccurlyeq y$ but not $y \preccurlyeq x$, and $x \sim y$, read as "y is indifferent to x" and defined by $x \sim y$ if and only if $x \preccurlyeq y$ and $y \preccurlyeq x$.

A preference ordering $\preccurlyeq$, hereafter simply *a preference*, defined on an ordered space $(X, \leqslant)$ is *monotone* if and only if for any two points x, $y \in X$, $x \leqslant y$ implies $x \preccurlyeq y$, and *strictly monotone* if and only if $x \leqslant y$ and $x \neq y$ imply $x \prec y$. The set of alternatives X is *convex* if a decision-maker who can choose between x and y can also choose a mixed strategy, i.e. a convex combination of both options. Formally, a preference $\preccurlyeq$ defined on a convex subset X of a linear space is said to be convex if and only if, for any two points x, $y \in X$ and $\lambda \in (0, 1)$, $x \preccurlyeq y$ implies $x \preccurlyeq \lambda x + (1 - \lambda)y$, and *strictly convex* if and only if $x \preccurlyeq y$ and $\lambda \in (0, 1)$ imply $x \prec \lambda x + (1 - \lambda)y$.[2]

When the set X is endowed with a topology τ, the preference $\preccurlyeq$ is *continuous* in (X, τ) if, for all $x \in X$, the sets $U_x = \{y \in X : x \preccurlyeq y\}$ and $L_x = \{y \in X : y \preccurlyeq x\}$ are τ-closed in X. Given that we assume $\preccurlyeq$ to be complete, this definition is equivalent to requiring that $\mathring{U}_x = \{y \in X : x \prec y\}$ and $\mathring{L}_x = \{y \in X : y \prec x\}$ are open sets for all $x \in X$.

A preference $\preccurlyeq$ defined on a set of alternatives X, has a utility representation (in short, is *representable*) if there exists a function $u : X \to \mathbb{R}$ such that, for all x, $y \in X$, $x \preccurlyeq y$ if and only if $u(x) \leqslant u(y)$. Such function u is called *utility function*. If there exists a continuous utility function which represents $\preccurlyeq$ defined on a topological space (X, τ), then $\preccurlyeq$ is said to be *continuously representable*. It is easy to check that every function $u : X \to \mathbb{R}$ defines a complete and transitive preference $\preccurlyeq$ on X such that $x \preccurlyeq y$ if and only if $u(x) \leqslant u(y)$. In the same way, $x \prec y$ if and only if $u(x) < u(y)$.

The existence of a utility function representing a preference $\preccurlyeq$ defined on a topological space (X, τ) is closely related with the properties of the space. A topological space (X, τ) is said to be *separable* if it contains a countable subset whose closure is X. That is, there exists a set $D = \{q_1, \ldots, q_n, \ldots\}$ such that $D \cap V \neq \varnothing$ for every non-empty open set V in X. The topological space (X, τ) is *second countable* (or alternatively, *perfectly separable*) if τ admits a countable basis of open sets. Every

[1] An exposition of the possibilities opened by these weakenings of rationality can be found in Sen (1970).

[2] If $\preccurlyeq$ is represented by a utility function u, these definitions correspond to quasi-concavity and strict quasi-concavity of the function u. For other definitions of convex preferences we refer to Debreu (1959).

perfectly separable topological space is separable, and every separable metric space is perfectly separable. A topological space (X, τ) is *connected* if there is no partition of X into two disjoint, non-empty closed sets. Also, X is *path-connected* if for all x, $y \in X$ there exists a continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$. Note that every path-connected space is connected and every convex set in a linear topological space is path-connected.

Regarding the assumption of completeness, it is worth noting the very interesting result by Schmeidler (1971), which shows that if a non-trivial preference $\preccurlyeq$ defined on a connected topological space X (non-trivial means that there are x and y on X such that $x \prec y$) is reflexive, transitive and the sets U_x and L_x are closed for all x, as well as $\mathring{U}_x$ and $\mathring{L}_x$ are open ($\preccurlyeq$ is continuous), then the preference $\preccurlyeq$ is complete.

3. Representation of preferences

In general, to guarantee the existence of a utility representation, both the preference and the space on which it is defined must satisfy certain conditions. In this Section we review some of the most relevant results in the study of continuous preferences' representability.

It is clear that the representation of a preference defined on any finite set of alternatives is trivial. But the existence of a continuous utility representation becomes relevant when we take the Euclidean space R^n as the commodity space, since it ensures the existence of a solution to the consumer's problem over any compact subset of the commodity space. The proof that a preference defined on a subset of the Euclidean space is representable is essentially the same as when the commodity space is any connected and separable topological space. On the other hand, when the space of alternatives is non-separable, we will see that there are continuous preferences without utility representation. Thus, in this case some extra assumptions are required to guarantee a utility representation. We will see that these assumptions on preferences, which ensure a utility representation, in addition frequently imply the existence of a solution to the consumer's problem.

3.1. The classroom example of representable preferences

A common method to build a utility function representing a given preference, usually taught in intermediate microeconomics courses or handbooks (e.g., Varian, 2010), is what we call *the classroom example*, showed in Figure 1.

The idea is conveniently intuitive. The consumption set $X = \{x \in \mathbb{R}^n; x \geqslant 0\}$ is the positive cone of the Euclidean space. In it, higher indifference curves represent more preferred consumption plans. As in Figure 1, one can see that the indifference curves intersect the diagonal line $\mathcal{D} = \{\bar{d} = (d, \ldots, d), d \geqslant 0\}$ exactly once. Then, given any point $x \in X$, let $I(x)$ be the *indifference curve* containing x and let us denote by $\bar{d}_x = (d_x, \ldots, d_x)$ the point in which $I(x)$ intersects with $\mathcal{D}$. Next, by labeling indifference curves with their distance from 0 to $\bar{d}_x$ and defining, say, $u(x) = \sqrt{n} d_x$, we obtain a utility function for the given preference. Please note that this is just one way to measure the distance from 0 to $\bar{d}_x$, and it would be possible to define the utility function u by any increasing function of d_x.

However, in this "naif" proof we implicitly assume that the preference is continuous and monotone. Moreover, it should be shown that the diagonal line actually intersects the indifference curves. Connectedness is a very good tool for this.

Let us consider a continuous and monotone preference relation $\preccurlyeq$ defined on the positive cone of

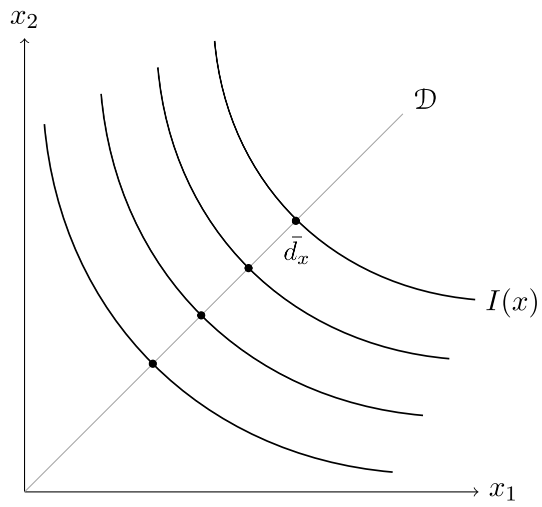

Figure 1: The classroom example

the Euclidean space. For all $x \in \mathbb{R}^n_+$, let us define

$$\begin{aligned} U_x &= \{\bar{d} \in \mathcal{D}; x \preccurlyeq \bar{d}\} \\ L_x &= \{\bar{d} \in \mathcal{D}; \bar{d} \preccurlyeq x\}. \end{aligned}$$

Note that U_x and L_x are closed sets in $\mathcal{D}$ because $\preccurlyeq$ is continuous, and that $U_x \cup L_x = \mathcal{D}$. If $\preccurlyeq$ is monotone then $U_x \neq \varnothing$ and $L_x \neq \varnothing$. Since $\mathcal{D}$ is connected, it must exist $\bar{d_x} = (d_x, \ldots, d_x) \in U_x \cap L_x$. That is, $\bar{d_x} \sim x$. It is easy to see that if we define $u(x) = u(\bar{d_x}) = d_x$, the function u represents $\preccurlyeq$. Indeed, $x \preccurlyeq y$ if and only if $\bar{d_x} \sim x \preccurlyeq y \sim \bar{d_y}$, and then $u(x) = d_x \leqslant d_y = u(y)$.

Note that this proof applies to any continuous preference $\preccurlyeq$ defined over any subset X of any Euclidean space containing a segment $\mathcal{D}'$ of the diagonal, where $\preccurlyeq$ is monotone on $\mathcal{D}'$ and has the following property: given any $x \in X$, there exist $\bar{d_x}$, $\bar{d^x} \in \mathcal{D}'$ such that $\bar{d_x} \preccurlyeq x \preccurlyeq \bar{d^x}$ (a property weaker than monotonicity). Further on we will see that, in fact, it applies to more general situations.

3.2. Example of a non-representable preference

The typical example of a preference without utility representation is the *lexicographic order* $\preccurlyeq_L$, defined in $[0,1] \times [0,1]$ by $(x_1, y_1) \preccurlyeq_L (x_2, y_2)$ if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 \leqslant x_2$. It is shown graphically in Figure 2.

Suppose that the lexicographic order $\preccurlyeq_L$ has a utility representation $u : [0,1] \times [0,1] \to \mathbb{R}$. Given any $x \in [0,1]$, since $(x, 0) \prec_L (x, 1)$, we must have $u(x, 0) < u(x, 1)$. Let us denote $A(x) = (u(x, 0), u(x, 1))$, a non-empty and open real interval that thus contains a rational number q_x. Note that $x \neq x'$ implies that $A(x) \cap A(x') = \varnothing$. Then, for each $x \in [0,1]$ we would have a rational number q_x, and as $A(x) \cap A(x') = \varnothing$, for $x \neq x'$ we would have $q_x \neq q'_x$. That is, we would have an injection of the reals into the rationals, which is impossible. Thus, $\preccurlyeq_L$ has no utility representation. To some extent, this is due the to the lack of enough real numbers, since any family of disjoint open intervals of real numbers must be finite or countable.

Note also that $\preccurlyeq_L$ is not a continuous preference in the space $[0,1] \times [0,1]$ endowed with the Euclidean topology. For it, consider any point (x, y) with $x < 1$ and $0 < y$, and the sequence $(x + \frac{1}{n}, \frac{y}{2})$. Observe that for all big enough n, $(x + \frac{1}{n}, \frac{y}{2}) \in U_{(x,y)} = \{(a, b); (x, y) \preccurlyeq_L (a, b)\}$.

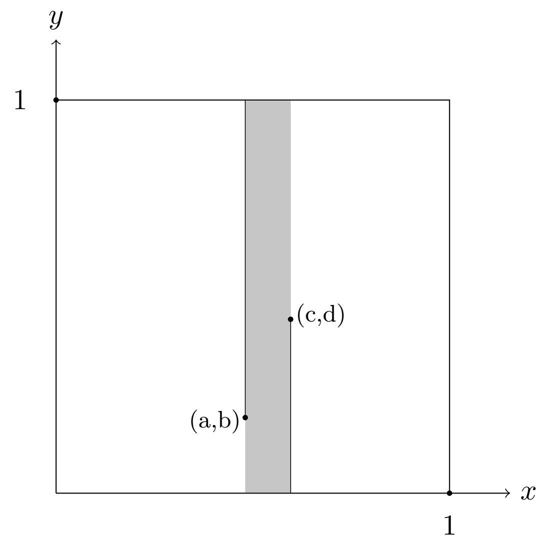

Figure 2: The lexicographic order

However, the limit of this sequence is $(x, \frac{y}{2})$, which is strictly less preferred than (x, y). Thus $U_{(x,y)}$ is not closed.

Indeed, this order could be defined in any subset X of any Euclidean space. Similar arguments such as those used before show that, except for irrelevant cases, $\preccurlyeq_L$ is neither continuous in the Euclidean topology nor representable by any utility function. Note that the lexicographic order $\preccurlyeq_L$ is, essentially, the only convex and monotonic preference defined on the Euclidean space that has no utility representation.[3]

3.3. General results for the existence of utility representation

Even if we were only interested in consumer theory, where preferences usually have particular properties like, for instance, convexity, local insatiability[4] or monotonicity, the most useful results on preference representation are very general.

The work of Debreu on the existence of continuous utility representation of preferences is what we call the "classical" existence result. Earlier contributions were performed by Eilenberg (1941), for a continuous strict total order in connected and separable spaces, and Wold (1943), who without explicitly assuming continuity listed a number of axioms (or conditions) a preference must meet in order to guarantee the existence of a real-valued utility representation. See also Mehta (1998).

Theorem 3.1 (Eilenberg-Debreu) Let (X, τ) be any topological space and $\preccurlyeq$ a continuous preference defined on X. If (X, τ) is either connected and separable or second countable, then there is a continuous utility function $u : X \to \mathbb{R}$ that represents the continuous preference $\preccurlyeq$.

For the proof, see Debreu (1954, 1964).

For connected and separable spaces the contribution of Debreu was to extend the previous result by Eilenberg from an order[5] to a preorder (a preference relation). Furthermore, his result for second countable topological spaces has the interesting feature that second countability is a hereditary

[3] We owe this observation to an anonymous referee.

[4] The preference $\preccurlyeq$ is locally insatiable if, for any point x in the consumption set X, and for any neighborhood V of x, there is another consumption $x' \in V$ such that $x \prec x'$.

[5] A preference relation $\preccurlyeq$ defined on X is an order on X if $x \preccurlyeq y$ and $y \preccurlyeq x$ implies $x = y$.

property, which in particular implies that any continuous preference defined on an arbitrary subset of any Euclidean space or any separable metric space is representable.

3.4. The need for separability

The proof by Debreu relies heavily on separability. For instance, in Debreu (1959)[6], in order to construct a utility representation u of a continuous preference $\preccurlyeq$ defined on a connected and separable space X (Debreu employed a Euclidean space), a countable dense set $D \subset X$ and the set of rational numbers $Q \subset [0,1]$ are considered. If the preference $\preccurlyeq$ has a least preferred point $x_m \in X$, then set $u(x_m) = 0$. If the preference has a most preferred point $x_M \in X$, then set $u(x_M) = 1$. Let us consider the points in D that are indifferent to x_M and to x_m, and let D' be the subset of D resulting from the elimination of these points. Then D' is countable and we can write it as $D' = \{x_1, x_2, \ldots, x_n, \ldots\}$, just like $Q' = \{q_1, q_2, \ldots, q_n, \ldots\} \cap (0,1)$. Next, Debreu defines an increasing function $u' : D' \to Q'$ in the following way. For x_1, define $u'(x_1) = q_1 = q_{r_1}$; for x_2, consider the partition of D': $(\leftarrow, x_1)$, $I(x_1) = \{x \in D'; x \sim x_1\}$, and $(x_1, \to)$. Now, with x_2 one of the following three cases must occur: $x_2 \sim x_1$, in which case take $r_2 = r_1$, and thus $u'(x_2) = q_{r_2} = q_{r_1}$; $x_2 \in (\leftarrow, x_1)$, in which case we consider the corresponding interval $(0, q_1) = (0, q_{r_1})$ contained in Q' and select the lowest integer number r_2, such that $q_{r_2} \in (0, q_{r_1})$, to then define $u'(x_2) = q_{r_2}$; or $x_2 \in (x_1, \to)$, a case in which we proceed in a similar way to the second case. Next, note that for x_3 we have to consider the classes of points which are indifferent either to x_1 or to x_2, as well as three possible intervals. For x_n we have to consider at most $n-1$ classes of indifference and n intervals. In all cases we proceed as with $u'(x_2)$. In this way, we will obtain a function $u' : D \to Q'$ that represents $\preccurlyeq$ on D. In order to extend u' to X consider any point $z \in X$, and define $L_z = \{x \in D; x \preccurlyeq z\}$ and $U_z = \{x \in D; z \preccurlyeq x\}$. If z is the least preferred point in X, set $u(z) = 0$, and if z is the most preferred point in X, set $u(x) = 1$. For any other z, let $u(z) = \sup\{u'(x); x \in L_z\} = \inf\{u'(x); x \in U_z\}$. This is an immediate consequence of the fact that u' takes all values of D'. Now, it is easy to see that u represents the preference $\preccurlyeq$, and thus, u is continuous.

It is worth mentioning that although Debreu was the first to make this proof in the utility theory setting, decades before Hausdorff (1914) had published an equivalent proof for order-preserving mappings from a connected ordered space to the real line.

For general topological spaces, separability seems to be a necessary property to guarantee the existence of a utility representation of any continuous preference. Notwithstanding, in many economic applications the set of alternatives could be non-separable. Consider an agent deciding over removable natural resources in an infinite temporal horizon, where the time of delivery is relevant. In this situation the decision-maker deals with infinitely many commodities, where the natural commodity space is l_∞, the space of bounded sequences of real numbers. When the state of the world (in the probabilistic sense) or the time of delivery are continuous variables we are faced with a similar scenario, in which the natural commodity space would be L_∞, the space of essentially bounded measurable functions defined on a measure space (see Bewley, 1972). Furthermore, Mas-Colell (1975) considers an infinite degree of commodity differentiation arguing that the natural commodity space is $ca(K)$, the space of countably additive signed measures over the compact space K. The spaces $ca(K)$, L_∞ and l_∞ are non-separable and so are their respective positive cones, which usually play the role of consumption sets.

On the other hand, recovering the example of the lexicographic preferences, it is clear that if we

[6] In this reference Debreu presents the results of his 1954's article in a more accessible way.

endow the square $X = [0,1] \times [0,1]$ with the topology of the lexicographic order, in which the basic open sets are the order open intervals $((a,b),\rightarrow),(\leftarrow,(c,d))$ and their intersections, then $\preccurlyeq_L$ is a continuous preference. The lack of a utility representation is due to the fact that $([0,1]\times[0,1],\preccurlyeq_L)$ is not separable. For this, given $x_1 < x_2 \in [0,1]$ and $x \in [x_1,x_2]$, consider the order interval $I_x = ((x,0),(x,1))$. Note that if $x \neq x'$, then $I_x \cap I_{x'} = \varnothing$. Let $D \subset [(x_1,y_1),(x_2,y_2)]$ be any dense set. As I_x is open, $I_x \cap D \neq \varnothing$, and therefore for each $x \in [x_1,x_2]$ a point $d_x \in D \cap I_x$ must exist. Note that $x \neq x'$ implies $d_x \neq d_{x'}$, which shows that D is uncountable.

Now, in order to stress the need for separability, we will see that in any non-separable metric space there are continuous preferences without utility representation.

Let us consider any non-separable metric space (X,d). Non-separability in metric spaces is characterized by the existence of an uncountable set $I \subset X$ and a real number $\epsilon > 0$ such that for every two different points x, $y \in I$, $d(x,y) > 2\epsilon$.

By using this property, Estévez-Toranzo and Hervés-Beloso (1995) define a preference $\preccurlyeq^*$ on X in which each point $x \in I$ is the most desirable point in its ϵ neighborhood; thus, any small perturbation of x is less preferred than x, and all other points outside the ϵ neighborhood of any point $x' \in I$ are even less desirable. It is shown, by using the *long line*[7], that $\preccurlyeq^*$ is continuous and has no utility representation.

Theorem 3.2 (Estévez-Toranzo and Hervés-Beloso, 1995, Theorem 1) Let X be any non-separable metric space. Then, there is a continuous preference on X which cannot be represented by a utility function.

The consequence of the result by Estévez-Toranzo and Hervés-Beloso (1995) is clear: in the case of non-separable spaces it is more general to consider preferences than utility functions.

However, if the commodity space E is an ordered non-separable Banach space, as the above-mentioned L_∞, l_∞ or $ca(K)$, or if it is $B([a,b])$ (the space of bounded functions defined on a real interval $[a,b]$, where $a < b$) and the consumption set X is a subset of E, the non-representable continuous preference $\preccurlyeq^*$ is neither monotonic nor convex. Yet, Monteiro (1987) provides an example of a convex, monotone and continuous preference defined on a closed convex subset of a Banach lattice that has no utility representation.

Next we will see that for some monotonic and continuous preferences it is still possible to get positive results in non-separable spaces.

3.5. Positive results in non-separable spaces

In order to consider monotonic preferences let us assume that the set of alternatives is a subset of an ordered linear space E. Although the commodity space might be non-separable, Mas-Colell (1986) shows[8] that the preference relation of every agent i, denoted by $\preccurlyeq_i$, has a utility representation over a consumption set contained in an order interval $[0,W]$.

For this, as in the "classroom example", let x be a point in $[0,W]$, take the real interval $[0,1]$ and denote $U_x = \{\lambda \in [0,1]; x \preccurlyeq_i \lambda W\}$ and $L_x = \{\lambda \in [0,1]; \lambda W \preccurlyeq_i x\}$, which, due to the continuity of $\preccurlyeq_i$, are closed sets. Observe that monotonicity of $\preccurlyeq_i$ implies that $W \in U_x$ and $0 \in L_x$. Also note that $L_x \cup U_x = [0,1]$, and that connectedness of the real interval implies $L_x \cap U_x \neq \varnothing$. Then, denote $\lambda_x \in L_x \cap U_x$ and define for consumer i the function $u_i : [0,W] \to \mathbb{R}$ by $u_i(x) = \lambda_x$. It is easy to see that u_i is a utility representation of $\preccurlyeq_i$.

[7]See Steen and Seebach (1970).

[8]See also Shafer (1984), who treats the utility representation of preferences defined in duals of normed spaces.

Monteiro (1987), inspired by Mas-Colell's result, provides a very useful representation theorem in path-connected spaces.

Definition 3.1 (Monteiro, 1987) A preference $\preccurlyeq$ defined on X is *countably bounded* if there exists a countable subset $D \subset X$ so that for every $x \in X$ there exist $d_x, d^x \in D$ such that $d_x \preccurlyeq x \preccurlyeq d^x$.

A subset $F \subset X$ bounds $\preccurlyeq$ if and only if for each $x \in X$ there exist $d_x, d^x \in F$ such that $d_x \preccurlyeq x \preccurlyeq d^x$. Therefore, $\preccurlyeq$ is countably bounded if it is bounded by a countable set $D \subset X$. Furthermore, it is easy to show that any preference with utility representation must be countably bounded.

Theorem 3.3 (Monteiro, 1987, Theorem 3) Let (X, τ) be path-connected. Any countably bounded continuous preference $\preccurlyeq$ defined on X has a continuous utility representation.

The proof relies on the separability of a connected set $F \subset X$ that bounds $\preccurlyeq$, and on the existence of a utility representation of $\preccurlyeq$ restricted to F (Theorem 3.1). It proceeds as follows: let $\{q_n; n \in \mathbb{N}\}$ be a countable set that bounds $\preccurlyeq$, and fix one point $x_0 \in X$. For each natural number n let $f_n : [0,1] \to X$ be a continuous function connecting x_0 with q_n; that is, $f_n(0) = x_0$ and $f_n(1) = q_n$. Define $F = \cup_n f_n([0,1])$ and note that it is path-connected, and therefore connected, as well as it is separable and that it bounds $\preccurlyeq$, since $q_n = f_n(1) \in F$. If we restrict $\preccurlyeq$ to F, by Theorem 3.1 there is a continuous utility function $u' : F \to \mathbb{R}$ such that for any points $a, b \in F, a \preccurlyeq b$ if and only if $u'(a) \leqslant u'(b)$.

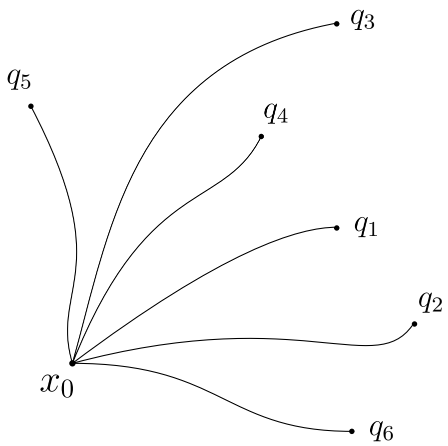

Figure 3: The construction of F

Now, the rest of the proof parallels the one shown in the previous section. Given any point $x \in X$, define the closed sets $L_x = \{a \in F; a \preccurlyeq x\}$ and $U_x = \{a \in F; x \preccurlyeq a\}$. We have that $L_x \cup U_x = F$ and, as $q_n \in F$ for all $n \in \mathbb{N}$, the sets L_x and U_x are non-empty. Therefore, by connectedness, it must exist $a_x \in F$ such that $a_x \sim x$. Then define $u : X \to \mathbb{R}$ by $u(x) = u'(a_x)$, which is a continuous utility representation of $\preccurlyeq$.

As an immediate corollary we have that if X is path-connected and $\preccurlyeq$ is continuous, then $\preccurlyeq$ has a utility representation if and only if it is countably bounded.

Note that if X is path-connected and the continuous preference $\preccurlyeq$ has a best $\bar{y}$ and a worst $\bar{x}$ point, that is, $\bar{x} \preccurlyeq x \preccurlyeq \bar{y}$ for all $x \in X$, then $\preccurlyeq$ has a continuous utility representation. Thus, the representation result by Mas-Colell is included in Monteiro's theorem. More generally, note that if X is a compact space and $\preccurlyeq$ is continuous, then $\preccurlyeq$ has a best $\bar{y}$ and a worst $\bar{x}$ point.

Therefore, in a compact and path-connected topological space every continuous preference has a utility representation.

One important case are the *σ-compact* spaces. A topological space X is σ-compact if there is a countable family of compact sets $\{K_n\}_n$ such that $\cup_n K_n = X$. Hence, if X is σ-compact and path-connected then any continuous preference $\preccurlyeq$ on X is countably bounded, and thus, it has a continuous utility representation. The set that bounds $\preccurlyeq$ is $D = \{(x_n, y_n)\}$, where x_n, y_n are, respectively, the best and worst points for $\preccurlyeq$ in the compact K_n.

The existence of utility representations of continuous preferences on subsets of σ-compact and path-connected spaces is relevant, for instance, when a decision-maker considers infinite time periods or states of nature. In those cases the commodity space is typically represented by a Banach space E and the set of alternatives is the positive cone, which is convex, and thus, path-connected. Note that E is σ-compact with the *weak** topology, and thus, Theorem 3.3 applies when $\preccurlyeq$ is continuous with this topology. The same result is true for subsets of normed spaces or *metrizable locally convex* spaces.

In particular, a weak* continuous preference defined on a convex subset of l_∞ or L_∞ is representable by a utility function. Note that the weak* continuity of the preference implies the countably boundedness, guaranteeing the existence of a utility representation. Note further that this condition also ensures the existence of a solution to the consumer's problem over any closed and bounded subset of the consumption set.

The key point in Monteiro's theorem is to obtain a connected and separable set that bounds $\preccurlyeq$. For that reason Candeal et al. (1998) define the following property:

Definition 3.2 (Candeal et al., 1998, Definition 2) A topological space (X, τ) is said to be *separably connected* if for every two points $x, y \in X$ there exists a connected and separable subset $C_{x,y} \subseteq X$ such that $x, y \in C_{x,y}$.

A separably connected space is connected since once we fix an element $x_0 \in X$, then $X = \cup_{x \in X} C_{x_0,x}$. On the other hand, it is clear that a path-connected topological space is separably connected because every path is connected and separable. However, not every separably connected space is path-connected. Consider the classic example $X = \{0\} \times [-1, 1] \cup \{(x, \sin(\frac{1}{x})) : x \in (0, 1]\}$ endowed with the Euclidean topology. It is well known that this space, represented in figure 4, is connected but not path-connected, while it is separably connected since it is connected and separable.

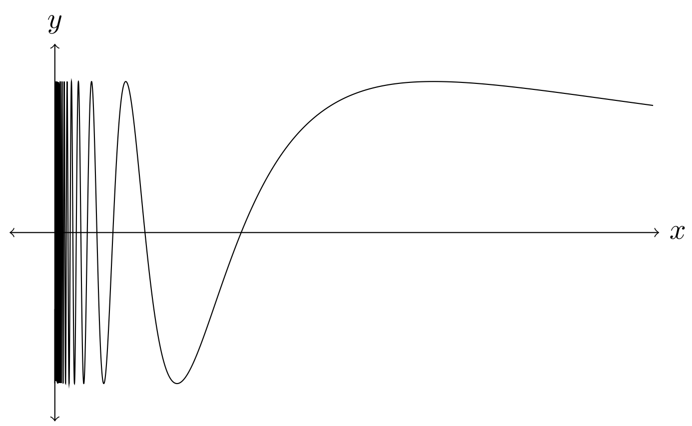

Figure 4: A connected space which is not path-connected

On the other hand, there are topological spaces that are connected but not separably connected. For example, consider, as we did in subsection 3.2, the lexicographic order $\preccurlyeq_L$ defined on the ordered topological space $X = [0,1] \times [0,1]$, with the order topology given by $\preccurlyeq_L$. This topological space is connected. The minimal connected set containing two points (a,b) and (c,d) with $a < c$ is the closed interval $[(a,b),(c,d)]$, which is not separable in this topology. Thus, $([0,1] \times [0,1], \preccurlyeq_L)$ is not separably connected.

Theorem 3.4 (Candeal et al., 1998, Theorem 4) Let (X, τ) be a separably connected space, and $\preccurlyeq$ a continuous preference defined on X. The preference $\preccurlyeq$ is representable if and only if it is countably bounded.

To see this, let $\{q_n; n \in \mathbb{N}\}$ be a countable set that bounds $\preccurlyeq$ and fix a point $x_0 \in X$. For each n, let C_{x_0,q_n} be a connected and separable subset of X containing both x_0 and q_n, and take $F = \cup_{n \in \mathbb{N}} C_{x_0,q_n}$, which is connected and separable. Now, the proof parallels the previous ones: we consider the restriction of $\preccurlyeq$ to F, which is continuous on F, and thus there exists a utility representation $u' : F \to \mathbb{R}$. Given any point $x \in X$, define the closed sets $L_x = \{a \in F; a \preccurlyeq x\}$ and $U_x = \{a \in F; x \preccurlyeq a\}$. As $q_n \in F$ for all n, F bounds $\preccurlyeq$, and thus, the sets L_x and U_x are non-empty, then $L_x \cup U_x = F$ and therefore it must exist $a_x \in F$ such that $a_x \sim x$. Next define $u : X \to \mathbb{R}$ by $u(x) = u'(a_x)$, which is a continuous utility representation of $\preccurlyeq$.

Since a preference represented by a utility function is countably bounded, we obtain the following Corollary (see Debreu, 1964):

Corollary 3.1 If a continuous preference defined on a separably connected space X has a utility representation, then it also has a continuous utility representation.

The notion of separable connectedness is useful to obtain a utility representation of a preference not covered by Monteiro's Theorem: let (X, τ) be a topological space and let $\{X_\alpha : \alpha \in A\}$ be a family of separably connected subsets of X, such that $X_\alpha \cap X_\beta \neq \varnothing$ for all $\alpha, \beta \in A$. Then $S = \cup_{\alpha \in A} X_\alpha$ is separably connected. Also, note that the countable cartesian product of separably connected topological spaces is also separably connected. Thus, the cartesian product of two separably connected factors, the first one being a non-path-connected space and the second one a non-separable space, gives an example of a separably connected space that is neither path-connected nor separable.

Suppose, for instance, that a consumer or a decision-maker has to choose in two time periods. In the first period she chooses over a connected and separable topological space X_1, whereas in the second period she does it over points in a convex subset of a non-separable Banach space X_2. Then, the consumption set is $X_1 \times X_2$, which may not be path-connected, but given that for any two points $a, b \in X$ there exists a connected and separable subset of X containing both points, X is separably connected.

For a more in-depth analysis of the property of separable connectedness, and to see more examples of non-path-connected but separably connected spaces, see Candeal et al. (1998) and Balbás de la Corte et al. (1998). In the particular case of metric spaces it is not easy to find examples of connected spaces which are not separably connected. In fact, Balbás de la Corte et al. (1998) conjectured that any connected set contained in any Banach space should be separably connected. However, the conjecture is false, as Aron and Maestre (2003) have shown; they proved that there are non-separable Banach spaces containing connected sets that are not separably connected. More recently Wójcik (2016) also addressed this conjecture.

4. Strictly monotonic preferences

A preference relation defined on a set of consumption plans is strictly monotonic whenever more consumption of at least one commodity is strictly preferred. This property is frequently assumed on agents' preferences. However, this Section shows that one should be cautious when the analysis involves a continuum of commodities.

Each commodity is characterized by its physical properties, its location, the time or date, the state of the world at the time of delivery, etc. (see Debreu, 1959, and Bewley, 1972). Thus, infinitely many commodities (a continuum of commodities) arise whenever one allows infinite variation in any of these characteristics.

Let K denote the set of commodities. Then, the commodity space is a subspace of the space $F(K)$ of the real functions defined on K, and a consumption set X is, in any case, a subset of $F(K)$. A consumption plan $f \in X$ is a function $f : K \to R$ that specifies an amount $f(k)$ of each commodity $k \in K$. Let us consider the standard partial order on $X \subset F(K)$; we write $f \leqslant g$ if $f(k) \leqslant g(k)$ for every $k \in K$, and $f < g$ if $f \leqslant g$ and $f \neq g$. For $f \leqslant g$ we define the order interval $[f, g] = \{h \in F(K); f \leqslant h \leqslant g\}$.

4.1. Results

An strictly monotonic preference is easily found on $X \subset F(K)$ if K is finite or countable. If K is finite X is a subset of the Euclidean space and the Cobb-Douglas preferences are strictly monotone. When K is countable, $F(K)$ is a sequences' space. Let us consider the case in which the consumption plans are bounded functions on K (bounded sequences). In this situation the commodity space is l_∞, the space of bounded sequences of real numbers. Given any sequence $\rho = (\rho_n)_{n\in\mathbb{N}}$, where $\rho_n > 0$ for all n, and $\sum_{n=1}^{\infty} \rho_n < +\infty$, the preference relation given by $a \preccurlyeq b$ if and only if $u(a) \leqslant u(b)$, where $u(x) = \sum_{n=1}^{\infty} \rho_n x_n$, is strictly monotone, continuous in both the norm and the weak* topology $\sigma(l_\infty, l_1)$, and, by definition, representable by the utility function u.

However, Theorem 1 in Hervés-Beloso and Monteiro (2010) shows that we cannot go much further. Consider an uncountable set of commodities, for example $K = [0, 1]$, or more generally the segment joining two points $a, b \in \mathbb{R}^n$, this is, $K = [a, b] = \{x; x = a+\lambda(b-a), \lambda \in [0, 1]\}$. Let A be any subset of K and denote with χ_A the characteristic function of the set A defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. We will assume that the consumption set X contains all the functions χ_A.

Formally, it is stated:

Assumption H: for any $A \subset K$, $\chi_A \in X$.

In some way Assumption H implies that the consumption set X is "rich enough" in relation to K. Given any set of commodities $A \subset K$, Assumption H implies that any agent could choose to consume one unit of each commodity of the subset A and none outside of it. An example of commodity space that fulfills Assumption H is the Banach space $B(K)$ of bounded functions on K:

$$B(K) = \{f : K \to \mathbb{R}; \sup\{|f(x)|, x \in K\} < \infty\}.$$

Theorem 4.1 (Hervés-Beloso and Monteiro, 2010, Theorem 1) Under Assumption H, every strictly monotonic preference relation on X is non-representable.

To prove it, let $\preccurlyeq$ be a strictly monotonic preference relation defined on X, and suppose that $u : X \to \mathbb{R}$ represents $\preccurlyeq$. Define for $x \in [a, b]$ the functions $f_x = \chi_{\{k\in K;k<x\}}$ and $f^x = \chi_{\{k\in K;k\leqslant x\}}$. Since $f_x < f^x$, by strict monotonicity we have that $f_x \prec f^x$, and therefore $u(f_x) < u(f^x)$. Now denote $I_x = (u(f_x), u(f^x))$ and notice that for $y \in [a,b]$ with $x < y$ we have that $f^x \prec f_y$, and therefore $I_y = (u(f_y), u(f^y)) \cap I_x = \varnothing$. Thus, we would obtain an uncountable family $\{I_x; x \in [a,b]\}$ of disjoint non-empty open real intervals, an impossibility.

We remark that this proof, in which K is, essentially, an interval of the real numbers, can easily be adapted to any uncountable set K. For this, we need to consider a *well-ordering* $\leqslant$ of K. It is said that $\leqslant$ is a well-order on K if for every non-empty subset $A \subset K$ there exists the minimum of A. Through Zermelo's theorem[9] we know that for every set K there is a well-ordering of K. Once we have considered a well-ordering of K, the proof for the general case is the same as Theorem 4.1's proof by considering that the role played by point 0 is now played by the minimum point with respect to the well-ordering of K.

In order to weaken Assumption H note that the proof of Theorem 4.1 only requires the existence of an uncountable subset $K_0 \subset K$ such that the restriction of the characteristic functions f_x and f^x to K_0 are in the consumption set X. Let u and v be bounded functions on K such that $u \leqslant v$ with $u(k) < v(k)$ for all k in the uncountable set K_0. The same proof would work if instead of supposing that X fulfills Assumption H we assume that the order interval $[u,v]$ is contained in X. For this, on the one hand define $f_x(k) = v(k)$ if $k < x$ and $f_x(k) = u(k)$ otherwise, and on the other hand $f^x(k) = v(k)$ if $k \leqslant x$ and $f^x(k) = u(k)$ otherwise. Note that the case $u(k) = 0$, $v(k) = 1$ for all k will give the characteristic function.

The next result shows that strictly monotonic preferences on any consumption set $X \subset F(K)$ always exists. We stress the importance of this result since, otherwise, Theorem 4.1 would be irrelevant.

Theorem 4.2 (Hervés-Beloso and Monteiro, 2010, Theorem 2) For any set K and for any consumption set $X \subset F(K)$, there exists a strictly monotonic preference relation on X.

To prove it, let $\leqslant_K$ be a well-ordering of K and $f, g \in X$, $f \neq g$. We denote $k^* = \min\{k \in K; f(k) \neq g(k)\}$. If $f(k^*) < g(k^*)$ we define $f \preccurlyeq g$. Otherwise we define $g \preccurlyeq f$. Note that $f \preccurlyeq g$ and $g \preccurlyeq f$ implies $f = g$. It is easy to verify that $\preccurlyeq$ defined in this way is complete, transitive and strictly monotonic.

Observe that the preference $\preccurlyeq$ just defined is a generalization of the lexicographic order, which is discontinuous in the natural topology of $B(K)$ and non-representable.

Let us restrict our attention to the important case where X is a convex subset of the positive cone of the space $B(K)$ and fulfills Assumption H. Suppose that τ is a topology such that $(B(K), \tau)$ is a Hausdorff linear space, and let $\preccurlyeq$ be any continuous and monotonic preference defined on X. Observe that if $\preccurlyeq$ is restricted to the order interval $[u,v]$, where $u, v \in B(K)$ are such that $u(k) < v(k)$ for all k, it is representable by a continuous utility function, since $\preccurlyeq$ is bounded in the convex set $[u,v]$. Thus, by Theorem 4.1 we conclude that $\preccurlyeq$ cannot be strictly monotonic[10].

This is, essentially, the proof of Hervés-Beloso and Monteiro (2010)'s last result.

Theorem 4.3 (Hervés-Beloso and Monteiro, 2010, Theorem 3) Let τ be a linear topology on the commodity space and suppose that the consumption set X fulfills Assumption H. Then, no strictly monotonic preference on X can be continuous on (X, τ).

[9]For a proof see Kelley (1955).

[10]Note that Assumption H guarantees the existence of the functions u and v.

4.2. Some Remarks

These results prove that strictly monotonic preferences always exist, but they are neither representable by utility functions nor continuous in any linear topology. This explains the lack of examples in the literature of strictly monotonic preferences defined on the Banach space $B(K)$ of the bounded real functions on an uncountable set K. Consequently we emphasize that one should be cautious when dealing with strictly monotonic preferences defined on a consumption set involving uncountably many commodities.

However, this caveat does not apply to the commodity space $L_\infty(K)$ of classes of essentially bounded measurable functions defined on K, as well as to the spaces of the classes of integrable functions on K. The reason is that the proof of Theorem 4.1 requires the functions $f_x = \chi_{\{k\in K; k<x\}}$ and $f^x = \chi_{\{k\in K; k\leqslant x\}}$ to be different elements, and provided that these functions only differ in x, both are the same element as class of functions on a σ-finite measure space. Another clear exception is the space $C(K)$ of continuous functions on K, since this space does not fulfill Assumption H.

Next, we provide a family of examples where there are uncountable many commodities.

Let $K = [0,1]$ be the set of commodities and let the consumption set X be the positive cone of the Banach space

$$l^p([0,1]) = \{f : [0,1] \to \mathbb{R}; \Sigma_{t\in[0,1]}|f(t)|^p < \infty\}.$$

For any $p, 1 \leqslant p < \infty$, there exist continuous and strictly monotonic preferences defined on X. For instance, let $u : l^p_+([0,1]) \to \mathbb{R}$ be defined by $u(f) = |f|_p = \Sigma_{t\in[0,1]}|f(t)|^p$. Then, the preference $\preccurlyeq$ defined by u is strictly monotonic, continuous and, obviously, representable by u. Note that this does not contradict the previous results since any $f \in l^p_+([0,1])$ is such that the set $\{t \in [0,1]; f(t) \neq 0\}$ is at most countable, and thus it does not fulfill Assumption H. Accordingly, these examples also show the necessity of Assumption H in Theorems 4.1 and 4.3.

Observe that any element f of any Banach space $l^p([0,1])$ is necessarily a bounded function, and thus, the consumption set X is also a subset of the space $B(K)$ of bounded functions on K. However, X is very narrow as a subset of $B(K)$ since any element $f \in X$ specifies the consumption of only a finite or countable number of commodities, or in other words, no alternative $f \in X$ involves uncountably many commodities.

In order to confront the incompatibility results with the case of continuous functions, the classes of integrable functions, let us precise that a preference is *purely* strictly monotonic whenever to consume more of at least one commodity is more preferred. In this regard, we emphasize that in the case of $C(K)$ or $L_\infty(K)$, the strict monotonicity is not pure since $f \leqslant g$ and $f \neq g$ imply that in the consumption plan g is consumed more than in plan f of an uncountable number of commodities.

5. Summing up

We have presented several relevant results in the literature of preference representation. These results guarantee the existence of utility functions representing continuous preferences defined over general consumption sets of commodity spaces that could be non-separable. In Table 1 we offer a summary of the Theorems discussed.

The results by Eilenberg (1941) and Debreu (1954, 1964) are very general, but rely heavily on separability. Estévez-Toranzo and Hervés-Beloso (1995) show that indeed, non-separable spaces might be problematic, since in any non-separable metric space there will always be a continuous

Table 1: Summary of Theorems

Eilenberg (1941) and Debreu (1954, 1964)	If the topological space (X, τ) is either connected and separable or second countable then there is a continuous utility function $u : X \to \mathbb{R}$ that represents the continuous preference $\preccurlyeq$.
Estévez-Toranzo and Hervés-Beloso (1995)	Let X be any non-separable metric space. Then, there is a continuous preference on X which cannot be represented by a utility function.
Monteiro (1987)	Let (X, τ) be path-connected. Any countably bounded continuous preference $\preccurlyeq$ defined on X has a continuous utility representation.
Candeal et al. (1998)	Let (X, τ) be a separably connected space, and $\preccurlyeq$ a continuous preference defined on X. The preference $\preccurlyeq$ is representable if and only if it is countably bounded.
Hervés-Beloso and Monteiro (2010)	For any set K and for any consumption set $X \subset F(K)$, there exists a strictly monotonic preference relation on X; and, If X fullfils Asumption H, then every strictly monotonic preference relation on X is neither continuous nor representable.

and non-representable preference.

Monteiro (1987) and Candeal et al. (1998) explore the existence of utility representation in non-separable spaces. In Monteiro (1987)'s paper the concept of countable boundedness is defined, and it is observed that any preference with utility representation must be countably bounded. Monteiro proves the existence of utility representation of any continuous and countably bounded preference defined on a path-connected space. Candeal et al. (1998), following Monteiro's idea, define the more general concept of separable connectedness and show that a continuous preference on such spaces is representable if and only if it is countably bounded.

Lastly, Hervés-Beloso and Monteiro (2010) analyze the special case of strictly monotonic preferences, showing that this fairly common assumption could be problematic if there is a continuum of commodities.

References

Aron, R.M. and Maestre, M. (2003) A connected metric space that is not serapably connected. *Contemporary Mathematics* 328: 39–42.

Balbás de la Corte, A., Estévez-Toranzo, M., Hervés-Beloso, C., and Verdejo Rodríguez, A. (1998) Espacios separablemente conexos. *Real Academia de Ciencias Exactas, Físicas y Naturales* 92(1): 35–40.

Bewley, T.F. (1972) Existence of equilibria in economies with infinitely many commodities. *Journal of Economic Theory* 4: 514–540.

Candeal, J.C., Hervés-Beloso, C., and Induráin, E. (1998) Some results on representation and extension of preferences. *Journal of Mathematical Economics* 29(1): 75–81.

Debreu, G. (1954). Representation of a preference ordering by a numerical function. Ed. by R. Thrall, R.L. Davis, and C. Coombs. *Decision processes.* (pp. 159–165). New York: Wiley.

Debreu, G. (1959) *Theory of Value: An Axiomatic Analysis of Economic Equilibrium.* New York: Wiley.

Debreu, G. (1964) Continuity Properties of Paretian Utility. *International Economic Review* 5(3): 285–293.

Eilenberg, S. (1941) Ordered topological spaces. *American Journal of Mathematics* 63(1): 39–45.

Estévez-Toranzo, M. and Hervés-Beloso, C. (1995) On the existence of continuous preference orderings without utility representations. *Journal of Mathematical Economics* 24(4): 305–309.

Fisher, I. (1892) Mathematical Investigations in the Theory of Value and Prices. *Transactions of the Connecticut Academy of Arts and Sciences* 9(I): 1–124.

Gordon, B.J. (1964) Aristotle and the Development of Value Theory. *The Quarterly Journal of Economics* 78(1): 115–128.

Hausdorff, F. (1914) *Grundzüge der Mengenlehre.* Leipzig: Veit.

Hervés-Beloso, C. and Monteiro, P.K. (2010) Strictly monotonic preferences on continuum of goods commodity spaces. *Journal of Mathematical Economics* 46(5): 725–727.

Hicks, J. and Allen, R.G.D. (1934) A Reconsideration of the Theory of Value (Parts 1 & 2). *Economica* NS 1(1; 2): 52–76, 196–219.

Kauder, E. (1953) Genesis of the Marginal Utility Theory: From Aristotle to the End of the Eighteenth Century. *The Economic Journal* 63(251): 638–650.

Kelley, J.L. (1955) *General Topology.* New York: Springer.

Lewin, S.B. (1996) Economics and Psychology: Lessons for Our Own Day From the Early Twentieth Century. *Journal of Economic Literature* 34(3): 1293–1323.

Mas-Colell, A. (1975) A model of equilibrium with differentiated commodities. *Journal of Mathematical Economics* 2(2): 263–295.

Mas-Colell, A. (1986) The Price Equilibrium Existence Problem in Topological Vector Lattices. *Econometrica* 54(5): 1039–1053.

Mehta, G.B. (1998). Preference and utility. Ed. by S. Barberà, P.J. Hammond, and C. Seidl. *Handbook of Utility Theory.* chap. 1: (pp. 1–47). Boston: Kluwer Academic Publishers.

Monteiro, P.K. (1987) Some results on the existence of utility functions on path connected spaces. *Journal of Mathematical Economics* 16(2): 147–156.

Pareto, V. (1900) Sul Fenomeno Economico: Lettera a Benedetto Croce. *Giornale degli Economisti* 21(2): 139–162. Reprinted as On the Economic Phenomenon: A Reply to Benedetto Croce. *International Economic Papers*, 1953, No. 3, 180–196.

Pareto, V. (1901) Sul Principio Economico: Lettera a Benedetto Croce. *Giornale degli Economisti* 22(2): 131–138. Reprinted as On the Economic Principle: A Reply to Benedetto Croce. *International Economic Papers*, 1953, No. 3, 203–207.

Pareto, V. (1906) *Manuale di economia politica.* Milan: Società Editrice Libraria. Reprinted as *Manual of Political Economy.* New York: Augustus M. Kelley, 1971.

Pareto, V. (1911). Economie Mathématique. Ed. by J. Molk. *Encyclopedie des Sciences Mathématiques.* 1(4, fasc. 4). (pp. 591–640). Paris: Gauthier-Villars. Reprinted as Mathematical Economics. *International Economic Papers*, 1955, No. 5, 58–102.

Samuelson, P.A. (1938) A Note on the Pure Theory of Consumer's Behaviour. *Economica* NS 5(17): 61–71.

Schmeidler, D. (1971) A Condition for the Completeness of Partial Preference Relations. *Econometrica* 39(2): 403–404.

Sen, A. (1970) *Collective Choice and Social Welfare*. San Francisco: Holden Day.

Shafer, W. (1984) *Representation of Preorders on Normed Spaces*. University of Southern California.

Slutsky, E. (1915) Sulla Teoria del Bilancio del Consumatore. *Giornale Degli Economisti* 51(1): 1–26.

Steen, L.A. and Seebach, J.A.J. (1970) *Counterexamples in Topology*. New York: Holt, Rinehart and Winston Inc.

Stigler, G.J. (1950) The Development of Utility Theory. I. *Journal of Political Economy* 58(4): 307–327.

Strotz, R.H. (1953) Cardinal Utility. *The American Economic Review* 43(2): 384–397.

Varian, H.R. (2010) *Intermediate microeconomics: a modern approach*. New York: WW Norton & Company.

Wójcik, M.R. (2016) The generalized Aron-Maestre Comb. *Houston Journal of Mathematics* 42(2): 701–707.

Wold, H. (1943) A Synthesis of Pure Demand Analysis; I, II and III. *Scandinavian Actuarial Journal* 26: 85–118, 220–263, 69–120.

CONTINUOUS PREFERENCE ORDERINGS REPRESENTABLE BY UTILITY FUNCTIONS

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Abstract. This article surveys the conditions under which it is possible to represent a continuous preference ordering using utility functions. We start with a historical perspective on the notions of utility and preferences, continue by defining the mathematical concepts employed in this literature, and then list several key contributions to the topic of representability. These contributions concern both, the preference orderings and the spaces where they are defined. For any continuous preference ordering, we show the need for separability and the sufficiency of connectedness and separability, or second-countability, of the space where it is defined. We emphasize the need of separability by showing that in any non-separable metric space there are continuous preference orderings without utility representation. However, by reinforcing connectedness, we show that countably boundedness of the preference ordering is a necessary and sufficient condition for the existence of a (continuous) utility representation. Lastly, we discuss the special case of strictly monotonic preferences.

Keywords. Preference ordering; Preference representation; Utility function

1. Introduction

The hypothesis that rational behavior can be represented as the maximization of a utility function is central to economic science. This article is about the conditions required for the existence of such representation. We aim to provide a summary of the main results on this topic which, despite its longevity, remains an area of active research. The introduction, even though this is a piece of mathematical economics, is devoted to offering a historical perspective on the notions of utility and preference.

The concept of *utility* commonly refers to the satisfaction received, or pain avoided, from consuming a good or service. Its usage can be at least traced back to Aristotle (Kauder, 1953; Gordon, 1964), and it was already employed extensively by Scholastics and Mercantilists, although it was not until the twentieth century that it acquired the meaning it has today. Up until the late nineteenth century economists understood utility as a measure of individuals' welfare, an attribute to rate human happiness. It was thought of as a quantifiable psychic magnitude, and a scale was even created for that matter, with "utils" being the name given to its units. Yet, how utility could actually be measured was never satisfactorily defined, even thoguh much attention was devoted to the task (most notably from the Marginalist school; see Stigler, 1950).

Towards the end of the Victorian era this understanding of utility started to be seen as vague and subjective, a notion not suitable for the scientific discipline that the economy was wanted to be. The lack of a "scientific" method to measure utility motivated some economists to change the approach, the first of these being Irving Fisher (1892) and Vilfredo Pareto (1900, 1901, 1906, 1911) (see also Lewin, 1996). These authors realized that with regard to utility, focusing on the *order* in which options are ranked —instead of on how much they differ from each other— served very well for the treatment of the problems to which utility theory was conventionally applied, and in this way utility no longer had to be thought of as a psychological entity measurable in its own right (Strotz, 1953). This approach is known as *ordinalism*, and paved the way for modern microeconomics; in fact, the resolvability of the demand problem is behind the ordinalist approach, and so is the representability problem. Then, instead of assessing the unobservable pleasure obtained (or pain avoided) from consuming a commodity, the object of study moved to the observable choice behavior between commodities, viz., the individuals' *preferences* (Varian, 2010). With this new focus, economic theory would no longer fall within the reign of psychics, trying to measure a subjective and intangible notion, but belong to the study of concrete, observable phenomena, just as physics did with the movement of bodies (Pareto, 1906).

The attempt to bring economics into scientific grounds was most enforced by the critical introduction of mathematical discourse, which would shape the discipline thereafter. Nevertheless, even though that much importance was given to it from the outset, it took decades until mathematics was properly applied to utility theory. Back when the move towards ordinalism took place it was blithely assumed that a *preference ordering* could always be measured numerically. This is, that it would always be possible to assign a real number to every consumption bundle available to a consumer in *representation* of the order in which she prefers them. This assigning is precisely what a *utility function* does, and is very helpful in solving the demand problem. Sadly, as we now know, it is not true that it can always be used to represent a preference ordering.

Building upon the works of Pareto, Eugene Slutsky (1915), John Hicks and Roy Allen (1934) and Paul Samuelson (1938), among others, contributed to make ordinalism the dominant approach in consumer theory, and explored the possibilities opened up by the study of preferences. Still, it was not until the work of Wold (1943) that due attention was paid to the conditions under which a utility function could represent some ranking of a consumer's preferences. The work of Wold was followed by that of Debreu and many more authors after him who, over the following decades and up to the present time, have been trying to refine and generalize the results on the matter. Reviewing this literature, regarding the conditions that guarantee a continuous utility representation of a preference ordering, is the object of this survey.

The paper is organized as follows: in Section 2 we present the technical definitions of the concepts employed. Section 3 starts with a basic example of how to represent a preference ordering, and continues with the typical counterexample of a non-representable preference. We use these cases to emphasize, in Subsection 3.3, the relevance of the assumptions of connectedness and separability (or second-countability) required by the "classical" existence result on preference representation. In Section 3.4 we point out that in any non-separable metric space there are continuous, transitive and complete preference orderings without utility representation. This could be problematic, as in many economic applications the consumption sets may be non-separable. This is the case, for instance, in the analysis of resource allocation over time or states of nature (Bewley, 1972) or in models exhibiting commodity differentiation (Mas-Colell, 1975). Nonetheless, in Subsection 3.5 we show that it is possible to overcome this problem by reinforcing connectedness of the space of alternatives and adding an assumption of boundedness of the continuous preference. In Section 4 we consider strictly monotonic preferences emphasizing the scenario where the set of commodities is uncountable. It is shown that strictly monotonic preference orderings always exist, and that if the commodity space is "rich enough" they cannot be continuous nor representable by any utility function. We conclude in Section 5 with a summary of the main results.

2. Definitions

A preference ordering, say $\precsim$, on a set X, is a *transitive* and *complete* (and thus, *reflexive*) binary relation on X. Hence, for every $x, y \in X$, either $x \precsim y$ or $y \precsim x$. Moreover, if $x \precsim y \precsim z$, then $x \precsim z$. We read $x \precsim y$ as "y is at least as preferred as x". In much of microeconomic theory individual preferences are assumed to be *rational*, with completeness and transitivity being necessary and sufficient conditions to guarantee this. However, weaker forms of rationality are possible, such as *quasitransitive* and *acyclic* preferences. Yet, since this paper is about preferences that can be represented by utility functions, which must be both transitive and complete, no weaker forms of rationality will be discussed.[1]

Two important relations derived from $\precsim$ are $x \prec y$, which reads "y is strictly preferred to x" and it is defined by $x \prec y$ if and only if $x \precsim y$ but not $y \precsim x$, and $x \sim y$, read as "y is indifferent to x" and defined by $x \sim y$ if and only if $x \precsim y$ and $y \precsim x$.

A preference ordering $\precsim$, hereafter simply *a preference*, defined on an ordered space $(X, \leqslant)$ is *monotone* if and only if for any two points x, $y \in X$, $x \leqslant y$ implies $x \precsim y$, and *strictly monotone* if and only if $x \leqslant y$ and $x \neq y$ imply $x \prec y$. The set of alternatives X is *convex* if a decision-maker who can choose between x and y can also choose a mixed strategy, i.e. a convex combination of both options. Formally, a preference $\precsim$ defined on a convex subset X of a linear space is said to be convex if and only if, for any two points x, $y \in X$ and $\lambda \in (0,1)$, $x \precsim y$ implies $x \precsim \lambda x + (1-\lambda)y$, and *strictly convex* if and only if $x \precsim y$ and $\lambda \in (0,1)$ imply $x \prec \lambda x + (1-\lambda)y$.[2]

When the set X is endowed with a topology τ, the preference $\precsim$ is *continuous* in (X, τ) if, for all $x \in X$, the sets $U_x = \{y \in X : x \precsim y\}$ and $L_x = \{y \in X : y \precsim x\}$ are τ-closed in X. Given that we assume $\precsim$ to be complete, this definition is equivalent to requiring that $\mathring{U}_x = \{y \in X : x \prec y\}$ and $\mathring{L}_x = \{y \in X : y \prec x\}$ are open sets for all $x \in X$.

A preference $\precsim$ defined on a set of alternatives X, has a utility representation (in short, is *representable*) if there exists a function $u : X \to \mathbb{R}$ such that, for all x, $y \in X$, $x \precsim y$ if and only if $u(x) \leqslant u(y)$. Such function u is called *utility function*. If there exists a continuous utility function which represents $\precsim$ defined on a topological space (X, τ), then $\precsim$ is said to be *continuously representable*. It is easy to check that every function $u : X \to \mathbb{R}$ defines a complete and transitive preference $\precsim$ on X such that $x \precsim y$ if and only if $u(x) \leqslant u(y)$. In the same way, $x \prec y$ if and only if $u(x) < u(y)$.

The existence of a utility function representing a preference $\precsim$ defined on a topological space (X, τ) is closely related with the properties of the space. A topological space (X, τ) is said to be *separable* if it contains a countable subset whose closure is X. That is, there exists a set $D = \{q_1, \dots, q_n, \dots\}$ such that $D \cap V \neq \varnothing$ for every non-empty open set V in X. The topological space (X, τ) is *second countable* (or alternatively, *perfectly separable*) if τ admits a countable basis of open sets. Every

[1] An exposition of the possibilities opened by these weakenings of rationality can be found in Sen (1970).

[2] If $\precsim$ is represented by a utility function u, these definitions correspond to quasi-concavity and strict quasi-concavity of the function u. For other definitions of convex preferences we refer to Debreu (1959).

perfectly separable topological space is separable, and every separable metric space is perfectly separable. A topological space (X, τ) is *connected* if there is no partition of X into two disjoint, non-empty closed sets. Also, X is *path-connected* if for all x, $y \in X$ there exists a continuous function $f : [0,1] \to X$ with $f(0) = x$ and $f(1) = y$. Note that every path-connected space is connected and every convex set in a linear topological space is path-connected.

Regarding the assumption of completeness, it is worth noting the very interesting result by Schmeidler (1971), which shows that if a non-trivial preference $\preccurlyeq$ defined on a connected topological space X (non-trivial means that there are x and y on X such that $x \prec y$) is reflexive, transitive and the sets U_x and L_x are closed for all x, as well as $\mathring{U}_x$ and $\mathring{L}_x$ are open ($\preccurlyeq$ is continuous), then the preference $\preccurlyeq$ is complete.

3. Representation of preferences

In general, to guarantee the existence of a utility representation, both the preference and the space on which it is defined must satisfy certain conditions. In this Section we review some of the most relevant results in the study of continuous preferences' representability.

It is clear that the representation of a preference defined on any finite set of alternatives is trivial. But the existence of a continuous utility representation becomes relevant when we take the Euclidean space R^n as the commodity space, since it ensures the existence of a solution to the consumer's problem over any compact subset of the commodity space. The proof that a preference defined on a subset of the Euclidean space is representable is essentially the same as when the commodity space is any connected and separable topological space. On the other hand, when the space of alternatives is non-separable, we will see that there are continuous preferences without utility representation. Thus, in this case some extra assumptions are required to guarantee a utility representation. We will see that these assumptions on preferences, which ensure a utility representation, in addition frequently imply the existence of a solution to the consumer's problem.

3.1. The classroom example of representable preferences

A common method to build a utility function representing a given preference, usually taught in intermediate microeconomics courses or handbooks (e.g., Varian, 2010), is what we call *the classroom example*, showed in Figure 1.

The idea is conveniently intuitive. The consumption set $X = \{x \in \mathbb{R}^n; x \geqslant 0\}$ is the positive cone of the Euclidean space. In it, higher indifference curves represent more preferred consumption plans. As in Figure 1, one can see that the indifference curves intersect the diagonal line $\mathcal{D} = \{\bar{d} = (d, \ldots, d), d \geqslant 0\}$ exactly once. Then, given any point $x \in X$, let $I(x)$ be the *indifference curve* containing x and let us denote by $\bar{d}_x = (d_x, \ldots, d_x)$ the point in which $I(x)$ intersects with $\mathcal{D}$. Next, by labeling indifference curves with their distance from 0 to $\bar{d}_x$ and defining, say, $u(x) = \sqrt{n} d_x$, we obtain a utility function for the given preference. Please note that this is just one way to measure the distance from 0 to $\bar{d}_x$, and it would be possible to define the utility function u by any increasing function of d_x.

However, in this "naif" proof we implicitly assume that the preference is continuous and monotone. Moreover, it should be shown that the diagonal line actually intersects the indifference curves. Connectedness is a very good tool for this.

Let us consider a continuous and monotone preference relation $\preccurlyeq$ defined on the positive cone of

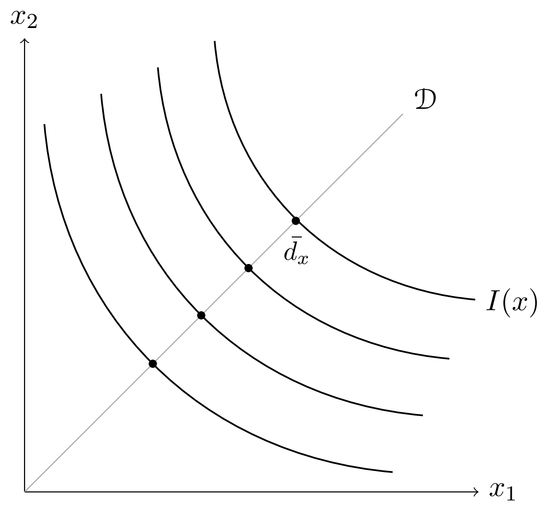

Figure 1: The classroom example

the Euclidean space. For all $x \in \mathbb{R}^n_+$, let us define

$$\begin{aligned} U_x &= \{\bar{d} \in \mathcal{D}; x \preccurlyeq \bar{d}\} \\ L_x &= \{\bar{d} \in \mathcal{D}; \bar{d} \preccurlyeq x\}. \end{aligned}$$

Note that U_x and L_x are closed sets in $\mathcal{D}$ because $\preccurlyeq$ is continuous, and that $U_x \cup L_x = \mathcal{D}$. If $\preccurlyeq$ is monotone then $U_x \neq \varnothing$ and $L_x \neq \varnothing$. Since $\mathcal{D}$ is connected, it must exist $\bar{d_x} = (d_x, \dots, d_x) \in U_x \cap L_x$. That is, $\bar{d_x} \sim x$. It is easy to see that if we define $u(x) = u(\bar{d_x}) = d_x$, the function u represents $\preccurlyeq$. Indeed, $x \preccurlyeq y$ if and only if $\bar{d_x} \sim x \preccurlyeq y \sim \bar{d_y}$, and then $u(x) = d_x \leqslant d_y = u(y)$.

Note that this proof applies to any continuous preference $\preccurlyeq$ defined over any subset X of any Euclidean space containing a segment $\mathcal{D}'$ of the diagonal, where $\preccurlyeq$ is monotone on $\mathcal{D}'$ and has the following property: given any $x \in X$, there exist $\bar{d_x}$, $\bar{d^x} \in \mathcal{D}'$ such that $\bar{d_x} \preccurlyeq x \preccurlyeq \bar{d^x}$ (a property weaker than monotonicity). Further on we will see that, in fact, it applies to more general situations.

3.2. Example of a non-representable preference

The typical example of a preference without utility representation is the *lexicographic order* $\preccurlyeq_L$, defined in $[0,1] \times [0,1]$ by $(x_1, y_1) \preccurlyeq_L (x_2, y_2)$ if either $x_1 < x_2$, or $x_1 = x_2$ and $y_1 \leqslant x_2$. It is shown graphically in Figure 2.

Suppose that the lexicographic order $\preccurlyeq_L$ has a utility representation $u : [0,1] \times [0,1] \to \mathbb{R}$. Given any $x \in [0,1]$, since $(x, 0) \prec_L (x, 1)$, we must have $u(x, 0) < u(x, 1)$. Let us denote $A(x) = (u(x, 0), u(x, 1))$, a non-empty and open real interval that thus contains a rational number q_x. Note that $x \neq x'$ implies that $A(x) \cap A(x') = \varnothing$. Then, for each $x \in [0,1]$ we would have a rational number q_x, and as $A(x) \cap A(x') = \varnothing$, for $x \neq x'$ we would have $q_x \neq q'_x$. That is, we would have an injection of the reals into the rationals, which is impossible. Thus, $\preccurlyeq_L$ has no utility representation. To some extent, this is due the to the lack of enough real numbers, since any family of disjoint open intervals of real numbers must be finite or countable.

Note also that $\preccurlyeq_L$ is not a continuous preference in the space $[0,1] \times [0,1]$ endowed with the Euclidean topology. For it, consider any point (x, y) with $x < 1$ and $0 < y$, and the sequence $(x + \frac{1}{n}, \frac{y}{2})$. Observe that for all big enough n, $(x + \frac{1}{n}, \frac{y}{2}) \in U_{(x,y)} = \{(a, b); (x, y) \preccurlyeq_L (a, b)\}$.

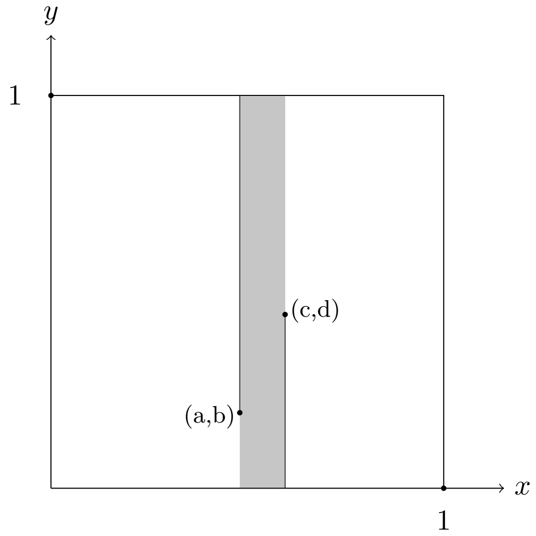

Figure 2: The lexicographic order

However, the limit of this sequence is $(x, \frac{y}{2})$, which is strictly less preferred than (x, y). Thus $U_{(x,y)}$ is not closed.

Indeed, this order could be defined in any subset X of any Euclidean space. Similar arguments such as those used before show that, except for irrelevant cases, $\preccurlyeq_L$ is neither continuous in the Euclidean topology nor representable by any utility function. Note that the lexicographic order $\preccurlyeq_L$ is, essentially, the only convex and monotonic preference defined on the Euclidean space that has no utility representation.[3]

3.3. General results for the existence of utility representation

Even if we were only interested in consumer theory, where preferences usually have particular properties like, for instance, convexity, local insatiability[4] or monotonicity, the most useful results on preference representation are very general.

The work of Debreu on the existence of continuous utility representation of preferences is what we call the "classical" existence result. Earlier contributions were performed by Eilenberg (1941), for a continuous strict total order in connected and separable spaces, and Wold (1943), who without explicitly assuming continuity listed a number of axioms (or conditions) a preference must meet in order to guarantee the existence of a real-valued utility representation. See also Mehta (1998).

Theorem 3.1 (Eilenberg-Debreu) Let (X, τ) be any topological space and $\preccurlyeq$ a continuous preference defined on X. If (X, τ) is either connected and separable or second countable, then there is a continuous utility function $u : X \to \mathbb{R}$ that represents the continuous preference $\preccurlyeq$.

For the proof, see Debreu (1954, 1964).

For connected and separable spaces the contribution of Debreu was to extend the previous result by Eilenberg from an order[5] to a preorder (a preference relation). Furthermore, his result for second countable topological spaces has the interesting feature that second countability is a hereditary

[3] We owe this observation to an anonymous referee.

[4] The preference $\preccurlyeq$ is locally insatiable if, for any point x in the consumption set X, and for any neighborhood V of x, there is another consumption $x' \in V$ such that $x \prec x'$.

[5] A preference relation $\preccurlyeq$ defined on X is an order on X if $x \preccurlyeq y$ and $y \preccurlyeq x$ implies $x = y$.

property, which in particular implies that any continuous preference defined on an arbitrary subset of any Euclidean space or any separable metric space is representable.

3.4. The need for separability

The proof by Debreu relies heavily on separability. For instance, in Debreu (1959)[6], in order to construct a utility representation u of a continuous preference $\preccurlyeq$ defined on a connected and separable space X (Debreu employed a Euclidean space), a countable dense set $D \subset X$ and the set of rational numbers $Q \subset [0,1]$ are considered. If the preference $\preccurlyeq$ has a least preferred point $x_m \in X$, then set $u(x_m) = 0$. If the preference has a most preferred point $x_M \in X$, then set $u(x_M) = 1$. Let us consider the points in D that are indifferent to x_M and to x_m, and let D' be the subset of D resulting from the elimination of these points. Then D' is countable and we can write it as $D' = \{x_1, x_2, \ldots, x_n, \ldots\}$, just like $Q' = \{q_1, q_2, \ldots, q_n, \ldots\} \cap (0,1)$. Next, Debreu defines an increasing function $u' : D' \to Q'$ in the following way. For x_1, define $u'(x_1) = q_1 = q_{r_1}$; for x_2, consider the partition of D': $(\leftarrow, x_1)$, $I(x_1) = \{x \in D'; x \sim x_1\}$, and $(x_1, \rightarrow)$. Now, with x_2 one of the following three cases must occur: $x_2 \sim x_1$, in which case take $r_2 = r_1$, and thus $u'(x_2) = q_{r_2} = q_{r_1}$; $x_2 \in (\leftarrow, x_1)$, in which case we consider the corresponding interval $(0, q_1) = (0, q_{r_1})$ contained in Q' and select the lowest integer number r_2, such that $q_{r_2} \in (0, q_{r_1})$, to then define $u'(x_2) = q_{r_2}$; or $x_2 \in (x_1, \rightarrow)$, a case in which we proceed in a similar way to the second case. Next, note that for x_3 we have to consider the classes of points which are indifferent either to x_1 or to x_2, as well as three possible intervals. For x_n we have to consider at most $n-1$ classes of indifference and n intervals. In all cases we proceed as with $u'(x_2)$. In this way, we will obtain a function $u' : D \to Q'$ that represents $\preccurlyeq$ on D. In order to extend u' to X consider any point $z \in X$, and define $L_z = \{x \in D; x \preccurlyeq z\}$ and $U_z = \{x \in D; z \preccurlyeq x\}$. If z is the least preferred point in X, set $u(z) = 0$, and if z is the most preferred point in X, set $u(x) = 1$. For any other z, let $u(z) = \sup\{u'(x); x \in L_z\} = \inf\{u'(x); x \in U_z\}$. This is an immediate consequence of the fact that u' takes all values of D'. Now, it is easy to see that u represents the preference $\preccurlyeq$, and thus, u is continuous.

It is worth mentioning that although Debreu was the first to make this proof in the utility theory setting, decades before Hausdorff (1914) had published an equivalent proof for order-preserving mappings from a connected ordered space to the real line.

For general topological spaces, separability seems to be a necessary property to guarantee the existence of a utility representation of any continuous preference. Notwithstanding, in many economic applications the set of alternatives could be non-separable. Consider an agent deciding over removable natural resources in an infinite temporal horizon, where the time of delivery is relevant. In this situation the decision-maker deals with infinitely many commodities, where the natural commodity space is l_∞, the space of bounded sequences of real numbers. When the state of the world (in the probabilistic sense) or the time of delivery are continuous variables we are faced with a similar scenario, in which the natural commodity space would be L_∞, the space of essentially bounded measurable functions defined on a measure space (see Bewley, 1972). Furthermore, Mas-Colell (1975) considers an infinite degree of commodity differentiation arguing that the natural commodity space is $ca(K)$, the space of countably additive signed measures over the compact space K. The spaces $ca(K)$, L_∞ and l_∞ are non-separable and so are their respective positive cones, which usually play the role of consumption sets.

On the other hand, recovering the example of the lexicographic preferences, it is clear that if we

[6] In this reference Debreu presents the results of his 1954's article in a more accessible way.

endow the square $X = [0,1] \times [0,1]$ with the topology of the lexicographic order, in which the basic open sets are the order open intervals $((a,b), \to), (\leftarrow, (c,d))$ and their intersections, then $\preccurlyeq_L$ is a continuous preference. The lack of a utility representation is due to the fact that $([0,1] \times [0,1], \preccurlyeq_L)$ is not separable. For this, given $x_1 < x_2 \in [0,1]$ and $x \in [x_1, x_2]$, consider the order interval $I_x = ((x,0),(x,1))$. Note that if $x \neq x'$, then $I_x \cap I_{x'} = \varnothing$. Let $D \subset [(x_1,y_1),(x_2,y_2)]$ be any dense set. As I_x is open, $I_x \cap D \neq \varnothing$, and therefore for each $x \in [x_1,x_2]$ a point $d_x \in D \cap I_x$ must exist. Note that $x \neq x'$ implies $d_x \neq d_{x'}$, which shows that D is uncountable.

Now, in order to stress the need for separability, we will see that in any non-separable metric space there are continuous preferences without utility representation.

Let us consider any non-separable metric space (X,d). Non-separability in metric spaces is characterized by the existence of an uncountable set $I \subset X$ and a real number $\epsilon > 0$ such that for every two different points x, $y \in I$, $d(x,y) > 2\epsilon$.

By using this property, Estévez-Toranzo and Hervés-Beloso (1995) define a preference $\preccurlyeq^*$ on X in which each point $x \in I$ is the most desirable point in its ϵ neighborhood; thus, any small perturbation of x is less preferred than x, and all other points outside the ϵ neighborhood of any point $x' \in I$ are even less desirable. It is shown, by using the *long line*[7], that $\preccurlyeq^*$ is continuous and has no utility representation.

Theorem 3.2 (Estévez-Toranzo and Hervés-Beloso, 1995, Theorem 1) Let X be any non-separable metric space. Then, there is a continuous preference on X which cannot be represented by a utility function.

The consequence of the result by Estévez-Toranzo and Hervés-Beloso (1995) is clear: in the case of non-separable spaces it is more general to consider preferences than utility functions.

However, if the commodity space E is an ordered non-separable Banach space, as the above-mentioned L_∞, l_∞ or $ca(K)$, or if it is $B([a,b])$ (the space of bounded functions defined on a real interval $[a,b]$, where $a < b$) and the consumption set X is a subset of E, the non-representable continuous preference $\preccurlyeq^*$ is neither monotonic nor convex. Yet, Monteiro (1987) provides an example of a convex, monotone and continuous preference defined on a closed convex subset of a Banach lattice that has no utility representation.

Next we will see that for some monotonic and continuous preferences it is still possible to get positive results in non-separable spaces.

3.5. Positive results in non-separable spaces

In order to consider monotonic preferences let us assume that the set of alternatives is a subset of an ordered linear space E. Although the commodity space might be non-separable, Mas-Colell (1986) shows[8] that the preference relation of every agent i, denoted by $\preccurlyeq_i$, has a utility representation over a consumption set contained in an order interval $[0,W]$.

For this, as in the "classroom example", let x be a point in $[0,W]$, take the real interval $[0,1]$ and denote $U_x = \{\lambda \in [0,1]; x \preccurlyeq_i \lambda W\}$ and $L_x = \{\lambda \in [0,1]; \lambda W \preccurlyeq_i x\}$, which, due to the continuity of $\preccurlyeq_i$, are closed sets. Observe that monotonicity of $\preccurlyeq_i$ implies that $W \in U_x$ and $0 \in L_x$. Also note that $L_x \cup U_x = [0,1]$, and that connectedness of the real interval implies $L_x \cap U_x \neq \varnothing$. Then, denote $\lambda_x \in L_x \cap U_x$ and define for consumer i the function $u_i : [0,W] \to \mathbb{R}$ by $u_i(x) = \lambda_x$. It is easy to see that u_i is a utility representation of $\preccurlyeq_i$.

[7]See Steen and Seebach (1970).

[8]See also Shafer (1984), who treats the utility representation of preferences defined in duals of normed spaces.

Monteiro (1987), inspired by Mas-Colell's result, provides a very useful representation theorem in path-connected spaces.

Definition 3.1 (Monteiro, 1987) A preference $\preccurlyeq$ defined on X is *countably bounded* if there exists a countable subset $D \subset X$ so that for every $x \in X$ there exist $d_x, d^x \in D$ such that $d_x \preccurlyeq x \preccurlyeq d^x$.

A subset $F \subset X$ bounds $\preccurlyeq$ if and only if for each $x \in X$ there exist $d_x, d^x \in F$ such that $d_x \preccurlyeq x \preccurlyeq d^x$. Therefore, $\preccurlyeq$ is countably bounded if it is bounded by a countable set $D \subset X$. Furthermore, it is easy to show that any preference with utility representation must be countably bounded.

Theorem 3.3 (Monteiro, 1987, Theorem 3) Let (X, τ) be path-connected. Any countably bounded continuous preference $\preccurlyeq$ defined on X has a continuous utility representation.

The proof relies on the separability of a connected set $F \subset X$ that bounds $\preccurlyeq$, and on the existence of a utility representation of $\preccurlyeq$ restricted to F (Theorem 3.1). It proceeds as follows: let $\{q_n; n \in \mathbb{N}\}$ be a countable set that bounds $\preccurlyeq$, and fix one point $x_0 \in X$. For each natural number n let $f_n : [0,1] \to X$ be a continuous function connecting x_0 with q_n; that is, $f_n(0) = x_0$ and $f_n(1) = q_n$. Define $F = \cup_n f_n([0,1])$ and note that it is path-connected, and therefore connected, as well as it is separable and that it bounds $\preccurlyeq$, since $q_n = f_n(1) \in F$. If we restrict $\preccurlyeq$ to F, by Theorem 3.1 there is a continuous utility function $u' : F \to \mathbb{R}$ such that for any points $a, b \in F, a \preccurlyeq b$ if and only if $u'(a) \leqslant u'(b)$.

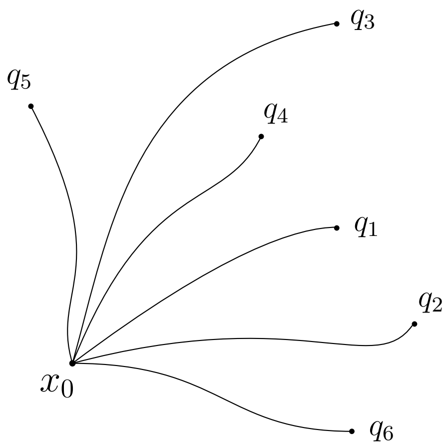

Figure 3: The construction of F

Now, the rest of the proof parallels the one shown in the previous section. Given any point $x \in X$, define the closed sets $L_x = \{a \in F; a \preccurlyeq x\}$ and $U_x = \{a \in F; x \preccurlyeq a\}$. We have that $L_x \cup U_x = F$ and, as $q_n \in F$ for all $n \in \mathbb{N}$, the sets L_x and U_x are non-empty. Therefore, by connectedness, it must exist $a_x \in F$ such that $a_x \sim x$. Then define $u : X \to \mathbb{R}$ by $u(x) = u'(a_x)$, which is a continuous utility representation of $\preccurlyeq$.

As an immediate corollary we have that if X is path-connected and $\preccurlyeq$ is continuous, then $\preccurlyeq$ has a utility representation if and only if it is countably bounded.

Note that if X is path-connected and the continuous preference $\preccurlyeq$ has a best $\bar{y}$ and a worst $\bar{x}$ point, that is, $\bar{x} \preccurlyeq x \preccurlyeq \bar{y}$ for all $x \in X$, then $\preccurlyeq$ has a continuous utility representation. Thus, the representation result by Mas-Colell is included in Monteiro's theorem. More generally, note that if X is a compact space and $\preccurlyeq$ is continuous, then $\preccurlyeq$ has a best $\bar{y}$ and a worst $\bar{x}$ point.

Therefore, in a compact and path-connected topological space every continuous preference has a utility representation.

One important case are the σ*-compact* spaces. A topological space X is σ-compact if there is a countable family of compact sets $\{K_n\}_n$ such that $\cup_n K_n = X$. Hence, if X is σ-compact and path-connected then any continuous preference $\preccurlyeq$ on X is countably bounded, and thus, it has a continuous utility representation. The set that bounds $\preccurlyeq$ is $D = \{(x_n, y_n)\}$, where x_n, y_n are, respectively, the best and worst points for $\preccurlyeq$ in the compact K_n.

The existence of utility representations of continuous preferences on subsets of σ-compact and path-connected spaces is relevant, for instance, when a decision-maker considers infinite time periods or states of nature. In those cases the commodity space is typically represented by a Banach space E and the set of alternatives is the positive cone, which is convex, and thus, path-connected. Note that E is σ-compact with the *weak** topology, and thus, Theorem 3.3 applies when $\preccurlyeq$ is continuous with this topology. The same result is true for subsets of normed spaces or *metrizable locally convex* spaces.

In particular, a weak* continuous preference defined on a convex subset of l_∞ or L_∞ is representable by a utility function. Note that the weak* continuity of the preference implies the countably boundedness, guaranteeing the existence of a utility representation. Note further that this condition also ensures the existence of a solution to the consumer's problem over any closed and bounded subset of the consumption set.

The key point in Monteiro's theorem is to obtain a connected and separable set that bounds $\preccurlyeq$. For that reason Candeal et al. (1998) define the following property:

Definition 3.2 (Candeal et al., 1998, Definition 2) A topological space (X, τ) is said to be *separably connected* if for every two points $x, y \in X$ there exists a connected and separable subset $C_{x,y} \subseteq X$ such that $x, y \in C_{x,y}$.

A separably connected space is connected since once we fix an element $x_0 \in X$, then $X = \cup_{x \in X} C_{x_0,x}$. On the other hand, it is clear that a path-connected topological space is separably connected because every path is connected and separable. However, not every separably connected space is path-connected. Consider the classic example $X = \{0\} \times [-1, 1] \cup \{(x, \sin(\frac{1}{x})) : x \in (0, 1]\}$ endowed with the Euclidean topology. It is well known that this space, represented in figure 4, is connected but not path-connected, while it is separably connected since it is connected and separable.

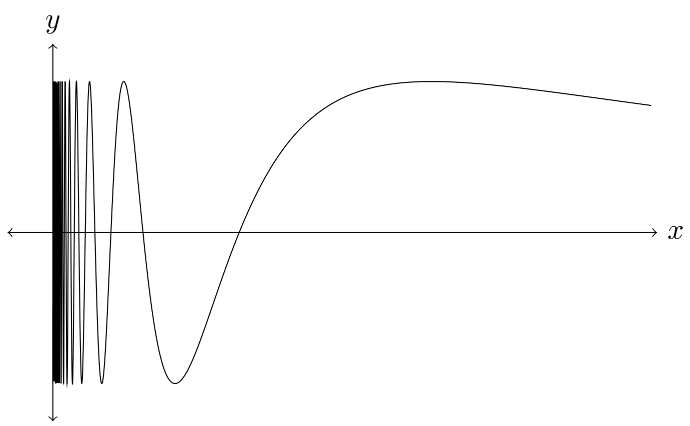

Figure 4: A connected space which is not path-connected

On the other hand, there are topological spaces that are connected but not separably connected. For example, consider, as we did in subsection 3.2, the lexicographic order $\preccurlyeq_L$ defined on the ordered topological space $X = [0,1] \times [0,1]$, with the order topology given by $\preccurlyeq_L$. This topological space is connected. The minimal connected set containing two points (a,b) and (c,d) with $a < c$ is the closed interval $[(a,b),(c,d)]$, which is not separable in this topology. Thus, $([0,1] \times [0,1], \preccurlyeq_L)$ is not separably connected.

Theorem 3.4 (Candeal et al., 1998, Theorem 4) Let (X,τ) be a separably connected space, and $\preccurlyeq$ a continuous preference defined on X. The preference $\preccurlyeq$ is representable if and only if it is countably bounded.

To see this, let $\{q_n; n \in \mathbb{N}\}$ be a countable set that bounds $\preccurlyeq$ and fix a point $x_0 \in X$. For each n, let C_{x_0,q_n} be a connected and separable subset of X containing both x_0 and q_n, and take $F = \cup_{n\in\mathbb{N}} C_{x_0,q_n}$, which is connected and separable. Now, the proof parallels the previous ones: we consider the restriction of $\preccurlyeq$ to F, which is continuous on F, and thus there exists a utility representation $u' : F \to \mathbb{R}$. Given any point $x \in X$, define the closed sets $L_x = \{a \in F; a \preccurlyeq x\}$ and $U_x = \{a \in F; x \preccurlyeq a\}$. As $q_n \in F$ for all n, F bounds $\preccurlyeq$, and thus, the sets L_x and U_x are non-empty, then $L_x \cup U_x = F$ and therefore it must exist $a_x \in F$ such that $a_x \sim x$. Next define $u : X \to \mathbb{R}$ by $u(x) = u'(a_x)$, which is a continuous utility representation of $\preccurlyeq$.

Since a preference represented by a utility function is countably bounded, we obtain the following Corollary (see Debreu, 1964):

Corollary 3.1 If a continuous preference defined on a separably connected space X has a utility representation, then it also has a continuous utility representation.

The notion of separable connectedness is useful to obtain a utility representation of a preference not covered by Monteiro's Theorem: let (X,τ) be a topological space and let $\{X_\alpha : \alpha \in A\}$ be a family of separably connected subsets of X, such that $X_\alpha \cap X_\beta \neq \varnothing$ for all $\alpha, \beta \in A$. Then $S = \cup_{\alpha\in A} X_\alpha$ is separably connected. Also, note that the countable cartesian product of separably connected topological spaces is also separably connected. Thus, the cartesian product of two separably connected factors, the first one being a non-path-connected space and the second one a non-separable space, gives an example of a separably connected space that is neither path-connected nor separable.

Suppose, for instance, that a consumer or a decision-maker has to choose in two time periods. In the first period she chooses over a connected and separable topological space X_1, whereas in the second period she does it over points in a convex subset of a non-separable Banach space X_2. Then, the consumption set is $X_1 \times X_2$, which may not be path-connected, but given that for any two points $a, b \in X$ there exists a connected and separable subset of X containing both points, X is separably connected.

For a more in-depth analysis of the property of separable connectedness, and to see more examples of non-path-connected but separably connected spaces, see Candeal et al. (1998) and Balbás de la Corte et al. (1998). In the particular case of metric spaces it is not easy to find examples of connected spaces which are not separably connected. In fact, Balbás de la Corte et al. (1998) conjectured that any connected set contained in any Banach space should be separably connected. However, the conjecture is false, as Aron and Maestre (2003) have shown; they proved that there are non-separable Banach spaces containing connected sets that are not separably connected. More recently Wójcik (2016) also addressed this conjecture.

4. Strictly monotonic preferences

A preference relation defined on a set of consumption plans is strictly monotonic whenever more consumption of at least one commodity is strictly preferred. This property is frequently assumed on agents' preferences. However, this Section shows that one should be cautious when the analysis involves a continuum of commodities.

Each commodity is characterized by its physical properties, its location, the time or date, the state of the world at the time of delivery, etc. (see Debreu, 1959, and Bewley, 1972). Thus, infinitely many commodities (a continuum of commodities) arise whenever one allows infinite variation in any of these characteristics.

Let K denote the set of commodities. Then, the commodity space is a subspace of the space $F(K)$ of the real functions defined on K, and a consumption set X is, in any case, a subset of $F(K)$. A consumption plan $f \in X$ is a function $f : K \to R$ that specifies an amount $f(k)$ of each commodity $k \in K$. Let us consider the standard partial order on $X \subset F(K)$; we write $f \leqslant g$ if $f(k) \leqslant g(k)$ for every $k \in K$, and $f < g$ if $f \leqslant g$ and $f \neq g$. For $f \leqslant g$ we define the order interval $[f, g] = \{h \in F(K); f \leqslant h \leqslant g\}$.

4.1. Results

An strictly monotonic preference is easily found on $X \subset F(K)$ if K is finite or countable. If K is finite X is a subset of the Euclidean space and the Cobb-Douglas preferences are strictly monotone. When K is countable, $F(K)$ is a sequences' space. Let us consider the case in which the consumption plans are bounded functions on K (bounded sequences). In this situation the commodity space is l_∞, the space of bounded sequences of real numbers. Given any sequence $\rho = (\rho_n)_{n\in\mathbb{N}}$, where $\rho_n > 0$ for all n, and $\sum_{n=1}^{\infty} \rho_n < +\infty$, the preference relation given by $a \preccurlyeq b$ if and only if $u(a) \leqslant u(b)$, where $u(x) = \sum_{n=1}^{\infty} \rho_n x_n$, is strictly monotone, continuous in both the norm and the weak* topology $\sigma(l_\infty, l_1)$, and, by definition, representable by the utility function u.

However, Theorem 1 in Hervés-Beloso and Monteiro (2010) shows that we cannot go much further. Consider an uncountable set of commodities, for example $K = [0, 1]$, or more generally the segment joining two points $a, b \in \mathbb{R}^n$, this is, $K = [a, b] = \{x; x = a+\lambda(b-a), \lambda \in [0, 1]\}$. Let A be any subset of K and denote with χ_A the characteristic function of the set A defined by $\chi_A(x) = 1$ if $x \in A$ and $\chi_A(x) = 0$ otherwise. We will assume that the consumption set X contains all the functions χ_A.

Formally, it is stated:

Assumption H: for any $A \subset K$, $\chi_A \in X$.

In some way Assumption H implies that the consumption set X is "rich enough" in relation to K. Given any set of commodities $A \subset K$, Assumption H implies that any agent could choose to consume one unit of each commodity of the subset A and none outside of it. An example of commodity space that fulfills Assumption H is the Banach space $B(K)$ of bounded functions on K:

$$B(K) = \{f : K \to \mathbb{R}; \sup\{|f(x)|, x \in K\} < \infty\}.$$

Theorem 4.1 (Hervés-Beloso and Monteiro, 2010, Theorem 1) Under Assumption H, every strictly monotonic preference relation on X is non-representable.

To prove it, let $\preccurlyeq$ be a strictly monotonic preference relation defined on X, and suppose that $u : X \to \mathbb{R}$ represents $\preccurlyeq$. Define for $x \in [a, b]$ the functions $f_x = \chi_{\{k\in K; k<x\}}$ and $f^x = \chi_{\{k\in K; k\leqslant x\}}$. Since $f_x < f^x$, by strict monotonicity we have that $f_x \prec f^x$, and therefore $u(f_x) < u(f^x)$. Now denote $I_x = (u(f_x), u(f^x))$ and notice that for $y \in [a,b]$ with $x < y$ we have that $f^x \prec f_y$, and therefore $I_y = (u(f_y), u(f^y)) \cap I_x = \varnothing$. Thus, we would obtain an uncountable family $\{I_x; x \in [a,b]\}$ of disjoint non-empty open real intervals, an impossibility.

We remark that this proof, in which K is, essentially, an interval of the real numbers, can easily be adapted to any uncountable set K. For this, we need to consider a *well-ordering* $\leqslant$ of K. It is said that $\leqslant$ is a well-order on K if for every non-empty subset $A \subset K$ there exists the minimum of A. Through Zermelo's theorem[9] we know that for every set K there is a well-ordering of K. Once we have considered a well-ordering of K, the proof for the general case is the same as Theorem 4.1's proof by considering that the role played by point 0 is now played by the minimum point with respect to the well-ordering of K.

In order to weaken Assumption H note that the proof of Theorem 4.1 only requires the existence of an uncountable subset $K_0 \subset K$ such that the restriction of the characteristic functions f_x and f^x to K_0 are in the consumption set X. Let u and v be bounded functions on K such that $u \leqslant v$ with $u(k) < v(k)$ for all k in the uncountable set K_0. The same proof would work if instead of supposing that X fulfills Assumption H we assume that the order interval $[u,v]$ is contained in X. For this, on the one hand define $f_x(k) = v(k)$ if $k < x$ and $f_x(k) = u(k)$ otherwise, and on the other hand $f^x(k) = v(k)$ if $k \leqslant x$ and $f^x(k) = u(k)$ otherwise. Note that the case $u(k) = 0$, $v(k) = 1$ for all k will give the characteristic function.

The next result shows that strictly monotonic preferences on any consumption set $X \subset F(K)$ always exists. We stress the importance of this result since, otherwise, Theorem 4.1 would be irrelevant.

Theorem 4.2 (Hervés-Beloso and Monteiro, 2010, Theorem 2) For any set K and for any consumption set $X \subset F(K)$, there exists a strictly monotonic preference relation on X.

To prove it, let $\leqslant_K$ be a well-ordering of K and $f, g \in X$, $f \neq g$. We denote $k^* = \min\{k \in K; f(k) \neq g(k)\}$. If $f(k^*) < g(k^*)$ we define $f \preccurlyeq g$. Otherwise we define $g \preccurlyeq f$. Note that $f \preccurlyeq g$ and $g \preccurlyeq f$ implies $f = g$. It is easy to verify that $\preccurlyeq$ defined in this way is complete, transitive and strictly monotonic.

Observe that the preference $\preccurlyeq$ just defined is a generalization of the lexicographic order, which is discontinuous in the natural topology of $B(K)$ and non-representable.

Let us restrict our attention to the important case where X is a convex subset of the positive cone of the space $B(K)$ and fulfills Assumption H. Suppose that τ is a topology such that $(B(K), \tau)$ is a Hausdorff linear space, and let $\preccurlyeq$ be any continuous and monotonic preference defined on X. Observe that if $\preccurlyeq$ is restricted to the order interval $[u,v]$, where $u, v \in B(K)$ are such that $u(k) < v(k)$ for all k, it is representable by a continuous utility function, since $\preccurlyeq$ is bounded in the convex set $[u,v]$. Thus, by Theorem 4.1 we conclude that $\preccurlyeq$ cannot be strictly monotonic[10].

This is, essentially, the proof of Hervés-Beloso and Monteiro (2010)'s last result.

Theorem 4.3 (Hervés-Beloso and Monteiro, 2010, Theorem 3) Let τ be a linear topology on the commodity space and suppose that the consumption set X fulfills Assumption H. Then, no strictly monotonic preference on X can be continuous on (X, τ).

[9]For a proof see Kelley (1955).

[10]Note that Assumption H guarantees the existence of the functions u and v.

4.2. Some Remarks

These results prove that strictly monotonic preferences always exist, but they are neither representable by utility functions nor continuous in any linear topology. This explains the lack of examples in the literature of strictly monotonic preferences defined on the Banach space $B(K)$ of the bounded real functions on an uncountable set K. Consequently we emphasize that one should be cautious when dealing with strictly monotonic preferences defined on a consumption set involving uncountably many commodities.

However, this caveat does not apply to the commodity space $L_\infty(K)$ of classes of essentially bounded measurable functions defined on K, as well as to the spaces of the classes of integrable functions on K. The reason is that the proof of Theorem 4.1 requires the functions $f_x = \chi_{\{k\in K; k<x\}}$ and $f^x = \chi_{\{k\in K; k\leqslant x\}}$ to be different elements, and provided that these functions only differ in x, both are the same element as class of functions on a σ-finite measure space. Another clear exception is the space $C(K)$ of continuous functions on K, since this space does not fulfill Assumption H.

Next, we provide a family of examples where there are uncountable many commodities.

Let $K = [0,1]$ be the set of commodities and let the consumption set X be the positive cone of the Banach space

$$l^p([0,1]) = \{f : [0,1] \to \mathbb{R}; \Sigma_{t\in[0,1]}|f(t)|^p < \infty\}.$$

For any $p, 1 \leqslant p < \infty$, there exist continuous and strictly monotonic preferences defined on X. For instance, let $u : l^p_+([0,1]) \to \mathbb{R}$ be defined by $u(f) = |f|_p = \Sigma_{t\in[0,1]}|f(t)|^p$. Then, the preference $\preccurlyeq$ defined by u is strictly monotonic, continuous and, obviously, representable by u. Note that this does not contradict the previous results since any $f \in l^p_+([0,1])$ is such that the set $\{t \in [0,1]; f(t) \neq 0\}$ is at most countable, and thus it does not fulfill Assumption H. Accordingly, these examples also show the necessity of Assumption H in Theorems 4.1 and 4.3.

Observe that any element f of any Banach space $l^p([0,1])$ is necessarily a bounded function, and thus, the consumption set X is also a subset of the space $B(K)$ of bounded functions on K. However, X is very narrow as a subset of $B(K)$ since any element $f \in X$ specifies the consumption of only a finite or countable number of commodities, or in other words, no alternative $f \in X$ involves uncountably many commodities.

In order to confront the incompatibility results with the case of continuous functions, the classes of integrable functions, let us precise that a preference is *purely* strictly monotonic whenever to consume more of at least one commodity is more preferred. In this regard, we emphasize that in the case of $C(K)$ or $L_\infty(K)$, the strict monotonicity is not pure since $f \leqslant g$ and $f \neq g$ imply that in the consumption plan g is consumed more than in plan f of an uncountable number of commodities.

5. Summing up

We have presented several relevant results in the literature of preference representation. These results guarantee the existence of utility functions representing continuous preferences defined over general consumption sets of commodity spaces that could be non-separable. In Table 1 we offer a summary of the Theorems discussed.

The results by Eilenberg (1941) and Debreu (1954, 1964) are very general, but rely heavily on separability. Estévez-Toranzo and Hervés-Beloso (1995) show that indeed, non-separable spaces might be problematic, since in any non-separable metric space there will always be a continuous

Table 1: Summary of Theorems

Eilenberg (1941) and Debreu (1954, 1964)	If the topological space (X, τ) is either connected and separable or second countable then there is a continuous utility function $u : X \to \mathbb{R}$ that represents the continuous preference $\preccurlyeq$.
Estévez-Toranzo and Hervés-Beloso (1995)	Let X be any non-separable metric space. Then, there is a continuous preference on X which cannot be represented by a utility function.
Monteiro (1987)	Let (X, τ) be path-connected. Any countably bounded continuous preference $\preccurlyeq$ defined on X has a continuous utility representation.
Candeal et al. (1998)	Let (X, τ) be a separably connected space, and $\preccurlyeq$ a continuous preference defined on X. The preference $\preccurlyeq$ is representable if and only if it is countably bounded.
Hervés-Beloso and Monteiro (2010)	For any set K and for any consumption set $X \subset F(K)$, there exists a strictly monotonic preference relation on X; and, If X fullfils Asumption H, then every strictly monotonic preference relation on X is neither continuous nor representable.

and non-representable preference.

Monteiro (1987) and Candeal et al. (1998) explore the existence of utility representation in non-separable spaces. In Monteiro (1987)'s paper the concept of countable boundedness is defined, and it is observed that any preference with utility representation must be countably bounded. Monteiro proves the existence of utility representation of any continuous and countably bounded preference defined on a path-connected space. Candeal et al. (1998), following Monteiro's idea, define the more general concept of separable connectedness and show that a continuous preference on such spaces is representable if and only if it is countably bounded.

Lastly, Hervés-Beloso and Monteiro (2010) analyze the special case of strictly monotonic preferences, showing that this fairly common assumption could be problematic if there is a continuum of commodities.

References

Aron, R.M. and Maestre, M. (2003) A connected metric space that is not serapably connected. *Contemporary Mathematics* 328: 39–42.

Balbás de la Corte, A., Estévez-Toranzo, M., Hervés-Beloso, C., and Verdejo Rodríguez, A. (1998) Espacios separablemente conexos. *Real Academia de Ciencias Exactas, Físicas y Naturales* 92(1): 35–40.

Bewley, T.F. (1972) Existence of equilibria in economies with infinitely many commodities. *Journal of Economic Theory* 4: 514–540.

Candeal, J.C., Hervés-Beloso, C., and Induráin, E. (1998) Some results on representation and extension of preferences. *Journal of Mathematical Economics* 29(1): 75–81.

Debreu, G. (1954). Representation of a preference ordering by a numerical function. Ed. by R. Thrall, R.L. Davis, and C. Coombs. *Decision processes.* (pp. 159–165). New York: Wiley.

Debreu, G. (1959) *Theory of Value: An Axiomatic Analysis of Economic Equilibrium.* New York: Wiley.

Debreu, G. (1964) Continuity Properties of Paretian Utility. *International Economic Review* 5(3): 285–293.

Eilenberg, S. (1941) Ordered topological spaces. *American Journal of Mathematics* 63(1): 39–45.

Estévez-Toranzo, M. and Hervés-Beloso, C. (1995) On the existence of continuous preference orderings without utility representations. *Journal of Mathematical Economics* 24(4): 305–309.

Fisher, I. (1892) Mathematical Investigations in the Theory of Value and Prices. *Transactions of the Connecticut Academy of Arts and Sciences* 9(I): 1–124.

Gordon, B.J. (1964) Aristotle and the Development of Value Theory. *The Quarterly Journal of Economics* 78(1): 115–128.

Hausdorff, F. (1914) *Grundzüge der Mengenlehre.* Leipzig: Veit.

Hervés-Beloso, C. and Monteiro, P.K. (2010) Strictly monotonic preferences on continuum of goods commodity spaces. *Journal of Mathematical Economics* 46(5): 725–727.

Hicks, J. and Allen, R.G.D. (1934) A Reconsideration of the Theory of Value (Parts 1 & 2). *Economica* NS 1(1; 2): 52–76, 196–219.

Kauder, E. (1953) Genesis of the Marginal Utility Theory: From Aristotle to the End of the Eighteenth Century. *The Economic Journal* 63(251): 638–650.

Kelley, J.L. (1955) *General Topology.* New York: Springer.

Lewin, S.B. (1996) Economics and Psychology: Lessons for Our Own Day From the Early Twentieth Century. *Journal of Economic Literature* 34(3): 1293–1323.

Mas-Colell, A. (1975) A model of equilibrium with differentiated commodities. *Journal of Mathematical Economics* 2(2): 263–295.

Mas-Colell, A. (1986) The Price Equilibrium Existence Problem in Topological Vector Lattices. *Econometrica* 54(5): 1039–1053.

Mehta, G.B. (1998). Preference and utility. Ed. by S. Barberà, P.J. Hammond, and C. Seidl. *Handbook of Utility Theory.* chap. 1: (pp. 1–47). Boston: Kluwer Academic Publishers.

Monteiro, P.K. (1987) Some results on the existence of utility functions on path connected spaces. *Journal of Mathematical Economics* 16(2): 147–156.

Pareto, V. (1900) Sul Fenomeno Economico: Lettera a Benedetto Croce. *Giornale degli Economisti* 21(2): 139–162. Reprinted as On the Economic Phenomenon: A Reply to Benedetto Croce. *International Economic Papers*, 1953, No. 3, 180–196.

Pareto, V. (1901) Sul Principio Economico: Lettera a Benedetto Croce. *Giornale degli Economisti* 22(2): 131–138. Reprinted as On the Economic Principle: A Reply to Benedetto Croce. *International Economic Papers*, 1953, No. 3, 203–207.

Pareto, V. (1906) *Manuale di economia politica.* Milan: Società Editrice Libraria. Reprinted as *Manual of Political Economy.* New York: Augustus M. Kelley, 1971.

Pareto, V. (1911). Economie Mathématique. Ed. by J. Molk. *Encyclopedie des Sciences Mathématiques.* 1(4, fasc. 4). (pp. 591–640). Paris: Gauthier-Villars. Reprinted as Mathematical Economics. *International Economic Papers*, 1955, No. 5, 58–102.

Samuelson, P.A. (1938) A Note on the Pure Theory of Consumer's Behaviour. *Economica* NS 5(17): 61–71.

Schmeidler, D. (1971) A Condition for the Completeness of Partial Preference Relations. *Econometrica* 39(2): 403–404.

Sen, A. (1970) *Collective Choice and Social Welfare*. San Francisco: Holden Day.

Shafer, W. (1984) *Representation of Preorders on Normed Spaces*. University of Southern California.

Slutsky, E. (1915) Sulla Teoria del Bilancio del Consumatore. *Giornale Degli Economisti* 51(1): 1–26.

Steen, L.A. and Seebach, J.A.J. (1970) *Counterexamples in Topology*. New York: Holt, Rinehart and Winston Inc.

Stigler, G.J. (1950) The Development of Utility Theory. I. *Journal of Political Economy* 58(4): 307–327.

Strotz, R.H. (1953) Cardinal Utility. *The American Economic Review* 43(2): 384–397.

Varian, H.R. (2010) *Intermediate microeconomics: a modern approach*. New York: WW Norton & Company.

Wójcik, M.R. (2016) The generalized Aron-Maestre Comb. *Houston Journal of Mathematics* 42(2): 701–707.

Wold, H. (1943) A Synthesis of Pure Demand Analysis; I, II and III. *Scandinavian Actuarial Journal* 26: 85–118, 220–263, 69–120.