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Efficiency and Endogenous Fertility*

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Abstract

This paper explores the properties of several notions of efficiency ($\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency and Millian efficiency) to evaluate allocations in a general overlapping generations setting with descendant altruism, in which fertility choices are endogenous and are selected from a continuum. We first focus on $\mathcal{A}-$efficiency, proposed by Golosov, Tertilt and Jones (*Econometrica*, 2007), and characterize $\mathcal{A}-$efficient allocations as those maximizing the utility of the dynasty head. With finite horizon altruism, this implies that $\mathcal{A}-$efficiency is characterized by the economy collapsing in finite time. To overcome this shortcoming, we propose to evaluate the efficiency of a given allocation with a particular specification of $\mathcal{P}-$efficiency –also proposed by Golosov *et al.*, 2007– for which the utility attributed to the unborn depends on the utility level achieved by those who get to be born in a given allocation. In regular economies, and for a large class of specifications of the function determining the utility attributed to the unborn, the class of symmetric, $\mathcal{P}-$efficient allocations coincides with the class of Millian efficient allocations; that is, the class of symmetric allocations that are not $\mathcal{A}-$dominated by any other symmetric allocation. Finally, we provide a version of the First Welfare Theorem by showing that, if a competitive equilibrium is symmetric, then *i*) it is statically Millian efficient; and, ii) if long-run wages do not exceed the capitalized costs of rearing children, then such symmetric competitive equilibrium is also –dynamically– Millian efficient and, hence, $\mathcal{P}-$efficient.

Key words: Efficiency, Optimal Population, Endogenous Fertility, $\mathcal{A}-$efficiency, $\mathcal{P}-$efficiency, Millian efficiency.

JEL: D91, H21, H5, E62, J13

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1 Introduction

The most commonly used optimality notion in standard normative economic analysis is that of Pareto efficiency. This notion of efficiency relies on the well-known Pareto criterion to compare social alternatives, a criterion that allows one to construct a *partial ordering* on a set of alternatives from the *complete preference orderings* (defined on this set) of a *fixed* group of agents. An efficient allocation can be described as a maximal element of the partial order induced by the Pareto criterion on the set of feasible allocations.

With endogenous fertility, one can still use the Pareto criterion to rank feasible allocations using the *partial orderings* of all potential agents, represented by the utility functions of the *living* agents. That is, an allocation can still be ranked as Pareto superior to another one if it is unanimously preferred by all potential agents according to their partial preference ordering. However, this implies that any two allocations with different fertility choices cannot be ranked, since there is no way to know whether or not an agent who lives in one allocation a but not in other allocation a' is better off in the latter than he is in the former. To avoid this problem and preserve the partial order induced by the Pareto criterion, one needs to extend it to compare also allocations of different fertility choices.

Although the issue seems to concern policymakers everywhere, the theoretical foundations of many proposals to alter fertility rates are rather weak. Most of the literature has simply turned to identify *optimal allocations* with the solutions to alternative social welfare maximization problems, referred to as *Millian* or *Benthamite* depending on whether or not the welfare weight given to a generation in the social welfare function depends on the size of that generation.[1] But this approach does not take into account the fact that the Pareto criterion is not directly applicable to environments in which the set of agents is endogenous. Besides, a social welfare maximization problem identifies a unique "optimal" allocation. As pointed out by Golosov, Jones and Tertilt (2007, p.1041), "such criteria (...) are very different in spirit from an efficiency concept that usually contains a larger number of allocations."

Unlike this literature, Lang (2005), Michel and Wigniolle (2007), and Conde-Ruiz *et al.* (2010) have provided normative principles to evaluate population policies in the context of an overlapping generations framework without altruism. These papers restrict their analysis to symmetric allocations, that is, to allocations in which any two agents of the same generation obtain the same consumption bundle. All these works focus on an extension of the Pareto criterion –which is referred to as *Representative Consumer dominance* by Michel *et al.* (2007) and as $\mathcal{A}-$*dominance* by Conde-Ruiz *et al.*– that ranks any two allocations of different population size by comparing exclusively the welfare profiles of those agents who are alive in the two allocations. Given the symmetry restriction on the set of allocations that are comparable using the $\mathcal{A}-$dominance criterion, a feasible symmetric allocation is said to be efficient (or, using the term proposed by Conde-Ruiz *et al.*, *Millian efficient*) if there does not exist an alternative feasible, symmetric allocation that provides all members of a generation with higher utility without decreasing the utility obtained by any other generation. The name "Millian" refers to the fact that it is a notion of efficiency that

[1] For a discussion on the different notions of optimality arising in settings with endogenous fertility, see e.g., Razin and Sadka (1995, Ch.5).

generalizes the notion of Millian optimality mentioned above.

But the restriction to symmetric allocations in environments without altruism limits the scope of all these papers. To fill this gap, Golosov, Jones and Tertilt (2007) have proposed two alternative extensions of the Pareto criterion. The first one is the $\mathcal{A}-$dominance criterion mentioned above, without restricting welfare comparisons to symmetric allocations. The second, referred to as the $\mathcal{P}-$dominance criterion, is constructed from a preliminary assumption on the utility level obtained by potential non-born agents. These two extensions of the notion of Pareto dominance give rise to two notions of efficiency, respectively referred to as $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, to evaluate allocations in environments in which fertility decisions are endogenous.

Golosov, Jones and Tertilt (henceforth GJT) provide partial characterizations of the two notions of efficiency as the solutions to welfare maximization problems, and prove that, under relatively mild assumptions, $\mathcal{A}-$efficient allocations are either $\mathcal{P}-$efficient or are arbitrarily close to allocations that are also $\mathcal{P}-$efficient (see GJT 2007, Sec.4.3 Result 3). Thus, the $\mathcal{P}-$efficiency of $\mathcal{A}-$efficient allocations is robust to different specifications of the utility levels attributed to the unborn. In a framework with *dynastic* altruism *à la* Barro and Becker (1989), GJT also explore the properties of a notion of equilibrium which results from the combination of the notion of competitive equilibria and the notion of subgame perfect equilibria of a transfer game played within families. In this environment, they provide a version of the First Welfare Theorem by showing that such equilibrium is both $\mathcal{A}-$efficient and $\mathcal{P}-$efficient (see GJT 2007, Th.2).

In this paper, we study the properties of the different notions of efficiency in a more general framework than GJT's extension of Barro and Becker's model. To be more precise, we focus on a two-period, overlapping generations setting in which *a*) the set of fertility choices is an unbounded interval in the positive real line; *b*) any two agents of the same generation who get to be alive have the same labor endowment and the same preferences, which depend on their own consumption of a homogeneous good, on the number of children they decide to bear and on the welfare obtained by their descendants. The setting covers, as particular cases, a wide range of positive models of fertility choice, including models in which altruism lies between the two polar representations (no altruism and dynastic altruism) considered mainly in the literature. The results of the paper can be gathered in three blocks.

On $\mathcal{A}-$efficiency. After describing the model and defining the different notions of efficiency in Section 3, we analyze the properties of $\mathcal{A}-$efficient allocations in Section 4. We show that, in all environments, only a small range of allocations can be regarded as $\mathcal{A}-$efficient: namely, those allocations that maximize the utility of the dynasty head (see Theorem 1). More specifically:

(*i*) *Concave value functions.* When value functions associated to dynastic maximization are concave, there exists, typically, a unique $\mathcal{A}-$efficient allocation, which is symmetric. Therefore, any other Millian efficient allocation is $\mathcal{A}-$inefficient, but it is not $\mathcal{A}-$dominated by the only $\mathcal{A}-$efficient allocation, which contrasts with Pareto efficiency in standard settings with complete information. There, taking any inefficient allocation as the *statu quo*, there are a wide range of allocations that are efficient and Pareto dominate the *statu quo*. As a particular case, when parents care only about consumption decisions of their immediate descendants, achieving $\mathcal{A}-$efficiency requires that a generation of agents must devote their entire endowment –or labor capacity– to provide with resources to their parents, which drives the economy to a collapse (Corollary 1).

(ii) *Non-concave value functions.* Since the set of feasible allocations is non-convex, value functions associated to dynastic optimization might be non concave, in which case $\mathcal{A}-$efficient allocations might be non-symmetric (Corollary 2). Although the set of $\mathcal{A}-$efficient allocations is not, in this case, a singleton, a similar result to that described in (i) occurs: there might exist many $\mathcal{A}-$inefficient allocations that are not $\mathcal{A}-$dominated by any $\mathcal{A}-$efficient allocation.

The intuition of why the notion of $\mathcal{A}-$efficiency reduces to dynastic maximization, which drives these results, is simple. Starting from any allocation a that does not maximize the utility of the dynasty head, it is always possible to find another allocation a' with more individuals that makes all people living under both a and a' better off than they were under a. Welfare improvements of this type (in the sense given by the $\mathcal{A}-$dominance criterion) can be achieved by enforcing every newcomer –that is, every individual living under a' who was not born under a– to use their endowment to maximize their parents' utilities. Newcomers can provide their parents with at least the same utility than those already living in a. Moreover, since some of the agents already living in a were not maximizing their parents' utilities, newcomers require fewer resources in order to achieve this objective. Finally, even though newcomers would rather prefer to obtain the same consumption bundles than those already living in a, this involves no welfare losses from the point of view of the $\mathcal{A}-$dominance criterion.

2. Millian efficiency as robust $\mathcal{P}-$efficiency. In view of the shortcomings (i) and (ii) faced by $\mathcal{A}-$efficiency and highlighted in the previous Section, we explore, in Section 5, whether or not the notion of $\mathcal{P}-$efficiency proposed by GJT is able to overcome these shortcomings. Differently from GJT, for whom the utility of the unborn is a cardinal value $\overline{u}$, we assume, instead, that the utility attributed to un unborn is a symmetric function of the utility achieved by the agent's living siblings. In this case, under certain (concavity) conditions, every symmetric allocation is $\mathcal{P}-$efficient if, and only if, it is Millian efficient (see Theorem 2). Furthermore, the $\mathcal{P}-$efficiency of Millian efficient allocations holds for a wide range of specifications of the utility attributed to the unborn. Finally, by assuming that the utility attributed to an unborn agent depends exclusively on the utility obtained by the agent's living siblings, we avoid cardinalist assessments in welfare comparisons.

3. Equilibrium Behavior. We conclude the paper by studying, in Section 6, the efficiency properties of a decentralized mechanism in which the agents, endowed with well-defined property rights over the commodities available in the economy, are free to trade these rights (or transfer them) to pursue their own interests. Differently from GJT, for whom transfers from parents to their children are unrestricted, we impose that transfers from parents to their children are non-negative (so that they are voluntarily accepted by the latter). We first show that, if value functions associated to the notion of competitive equilibrium are concave on a certain range, then competitive equilibria are symmetric. Then, we show in Theorem 3 that i) a symmetric competitive equilibrium is a –statically– Millian efficient allocation; and, ii) if long-run wages do not exceed the capitalized costs of rearing children, then a competitive equilibrium is also –dynamically– Millian efficient and, in view of our previous results, $\mathcal{P}-$efficient. Thus, for the notions of Millian efficiency and $\mathcal{P}-$efficiency, a version of the First Welfare Theorem holds. Therefore, when applied to Millian efficiency or $\mathcal{P}$-efficiency, potential markets failures are of the same nature as those affecting Pareto efficient allocations in dynamic economies with exogenous fertility: although competitive equilibria are always statically efficient –i.e, it cannot be improved upon by a reallocation of the resources available for a finite number of generations–, they might be inefficient (or dynamically inefficient) –that is, they can be improved upon by a reallocation of the resources available of all generations.

The main conclusions of the paper and further research are discussed in Section 7.

2 Notation and feasible allocations

We consider a particular class of overlapping generations economies with infinite periods of time in which each individual lives for at most three of these periods, so that individuals living at $t = 0, 1, 2,...$ are referred to as *children, middle-aged adults* or *old adults* depending on whether t is their first, their second or their third period of life.

As in GJT, the set of potential agents that are actually alive at any given period is endogenous and it depends on fertility plans selected by the agents. For $t = 0, 1, 2,...$, the set of possible fertility choices available to every middle-aged adult is $\mathbb{R}_+$, and the set of potential middle-aged agents at period t is $\mathbb{R}^t_+$. Each middle-aged adult *potentially* alive at $t = 1$ is identified by a positive number $i_1 \in \mathbb{R}_+$ determining the agent's position in the sibling order. For $t = 2, 3,...$, each middle-aged adult potentially alive at t is identified by a vector $i^t = (i^{t-1}, i_t) \in \mathbb{R}^t_+$, where i_t specifies the agent's position in the sibling order, and $i^{t-1} = (i_1, ..., i_{t-1})$ identifies the agent's parent. To simplify things, all agents belong to the same dynasty, initiated by the only agent who is middle aged at $t = 0$, the *dynasty head*, hereafter represented by i^0. Let $\mathcal{B}(\mathbb{R}^t_+)$ the class of Borelian sets in $\mathbb{R}^t_+$. For every set $B^t \in \mathcal{B}(\mathbb{R}^t_+)$ of potential middle-aged agents at t, the (Lebesgue) measure of B^t will be denoted by $\mu_L \{B^t\} \equiv \int_{B^t} di^t$.

A fertility plan n is a sequence of integrable functions $\mathsf{n} = \{\mathsf{n}_{t+1} : \mathbb{R}^t_+ \longrightarrow \mathbb{R}_+\}_{t\geq 0}$ that determines, for every t and every $i^t \in \mathbb{R}^t_+$, the number of descendants $\mathsf{n}_{t+1}(i^t)$ that agent i^t decides to have during her second period of life. Hence, for each t and every $i^t = (i^{t-1}, i_t) \in \mathbb{R}^t_+$, agent i^t is said to be alive with fertility plan n if agent i^{t-1} is also alive and $i_t \leq \mathsf{n}_t(i^{t-1})$ is satisfied. For every individual $i^t \in \mathbb{R}^t_+$ and every $\tau \geq t + 1$, the set of descendants of i^t belonging to generation τ is denoted by $\mathcal{D}_\tau(i^t)$. The set of middle-aged adults actually living at t with a fertility plan n is denoted by $\mathcal{I}_t(\mathsf{n})$ and its measure, denoted by $\mathcal{N}_t(\mathsf{n})$, is given by

$$\mathcal{N}_t(\mathsf{n}) = \mu_L \{\mathcal{I}_t(\mathsf{n})\} = \int_{\mathcal{I}_t(\mathsf{n})} di^t = \int_{\mathcal{I}_{t-1}(\mathsf{n})} \left(\int_{i_t \leq \mathsf{n}_t(i^{t-1})} di_t \right) di^{t-1} = \int_{\mathcal{I}_{t-1}(\mathsf{n})} \mathsf{n}_t(i^{t-1}) di^{t-1}.$$

With respect to the set of commodities, in addition to children, there is only one homogenous good produced at every period $t \geq 1$. This consumption good is produced at each period $t = 0, 1, 2, ...,$ using physical capital (K_t), i.e. the amount of the same good invested in the previous period $t - 1$, and labor (L_t) provided by middle-aged adults as inputs; that is, $Y_t \leq F_t(K_t, L_t)$, where Y_t is total output and $F_t : \mathbb{R}^2_+ \longrightarrow \mathbb{R}_+$ exhibits constant returns to scale and it is non-decreasing, concave and continuously differentiable.

Rearing children is a production activity that takes place within each household and its costs are represented by a strictly increasing, convex and continuously differentiable function $b_t : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$. Thus, a middle-aged adult who decides to rear n_{t+1} children at period t needs to spend $b_t(n_{t+1})$ units of the consumption good. Fertility and consumption plans of potential agents are represented by a fertility plan n and a sequence of integrable functions $\mathsf{c} = \left\{(\mathsf{c}^m_t, \mathsf{c}^o_{t+1}) : \mathbb{R}^t_+ \longrightarrow \mathbb{R}^2_+\right\}_{t\geq 0}$ that determines, for each $t \geq 0$ and each potential agent $i^t \in \mathbb{R}^t_+$, the consumption vector $(\mathsf{c}^m_t(i^t), \mathsf{c}^o_{t+1}(i^t))$ chosen by agent i^t through her life cycle. Thus, it is assumed that children do not take consumption decisions.

The resource constraint faced by potential agents is described as follows. At time $t = 0$, the amount of resources available to finance consumption $(\mathsf{c}^m_0(i^o))$, fertility $(\mathsf{n}_1(i^o))$ and investment decisions $(\mathsf{k}^o_1(i^o))$ of the dynasty head is bounded by an initial endowment $\overline{e}_0$ available for the dynasty head, that is,

$$\mathsf{c}^m_0(i^o) + b_0(\mathsf{n}_1(i^o)) + \mathsf{k}^o_1(i^o) \leq \overline{e}_0. \tag{1}$$

For each period $t \geq 0$, each agent who gets to be alive is endowed with 1 unit of labor time when she reaches her middle age. Then, labor is supplied inelastically, so that labor supply at any given period coincides with the measure of middle-aged agents alive at t, that is, $L_t = \mathcal{N}_t(\mathsf{n})$. By writing, for each t and each $i^t \in \mathcal{I}_t(\mathsf{n})$, $\mathsf{k}^o_{t+1}(i^t)$ for $\mathsf{k}^o_{t+1}(i^t) = \mathsf{n}_{t+1}(i^t)\frac{K_{t+1}}{\mathcal{N}_t(\mathsf{n})}$,the resource constraint at each date $t \geq 1$ is

$$\int_{\mathcal{I}_{t-1}(\mathsf{n})} \mathsf{c}^o_t(i^{t-1})di^{t-1} + \int_{\mathcal{I}_t(\mathsf{n})} \left[\mathsf{c}^m_t(i^t) + b_t(\mathsf{n}_{t+1}(i^t)) + \mathsf{k}^o_{t+1}(i^t)\right] di^t \leq \int_{\mathcal{I}_{t-1}(\mathsf{n})} F_t(\mathsf{k}^o_t(i^{t-1}), \mathsf{n}_t(i^{t-1}))di^{t-1}. \tag{2}$$

In what follows, an allocation is a pair $\mathsf{a} = (\mathsf{x}, \mathsf{k}^o) \in \mathcal{X} \times \mathcal{K}$, where $\mathcal{X}$ is the set of sequences of the form $\mathsf{x} = \left\{\mathsf{x}_t = (\mathsf{c}^m_t, \mathsf{c}^o_{t+1}, \mathsf{n}_{t+1}) : \mathbb{R}^t_+ \longrightarrow \mathbb{R}^3_+\right\}_{t\geq 0}$ determining consumption and fertility choices of every potential agent and $\mathcal{K}$ is the set of sequences of the form $\mathsf{k}^o = \left\{\mathsf{k}^o_{t+1} : \mathbb{R}^t_+ \longrightarrow \mathbb{R}_+\right\}_{t\geq 0}$ determining investment decisions in every period. Without loss of generality, it is also assumed that for every t and every $i^t \in \mathbb{R}^t$ one has

$$\mathsf{x}_t(i^t) = 0 \text{ and } \mathsf{k}^o_{t+1}(i^t) = 0 \text{ whenever } i^t \notin \mathcal{I}_t(\mathsf{n}). \tag{3}$$

An allocation a is feasible if it satisfies the initial condition in (1), the resource constraint in (2), as well as condition (3). The set formed by all feasible allocations is denoted by $\mathcal{F} \subset \mathcal{X} \times \mathcal{K}$.

Write $\mathbb{R}^*$ for the set of extended real numbers $\mathbb{R}^* \equiv \{-\infty\} \cup \mathbb{R}$. Throughout the paper, we assume that preferences of every potential agent of generation t on the set of allocations in which the agent is alive are represented by a utility function $\mathcal{U}_t : \mathcal{X} \times \mathbb{R}^t_+ \longrightarrow \mathbb{R}^*$ satisfying, for every $\mathsf{x} \in \mathcal{X}$ and $i^t \in \mathcal{I}_t(\mathsf{n})$

$$\mathcal{U}_t(\mathsf{x}; i^t) = U\left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_0^{\mathsf{n}_{t+1}(i^t)} \mathcal{U}^D_{t+1}(\mathsf{x}; i^t, i_{t+1})di_{t+1}\right),$$

where $U : \mathcal{X} \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ is a non-decreasing function and $\mathcal{U}^D_{t+1}$ is a function representing the agent's preferences on fertility and consumption choices made by her living descendants.

Specifically, the welfare that any agent i^t obtains from consumption decisions of all his descendants born after t is represented by a utility function $\mathcal{U}_t^D : \mathcal{X} \times \mathbb{R}_+^t \longrightarrow \mathbb{R}$ satisfying, for every $\mathsf{x} \in \mathcal{X}$ and $i^t \in \mathcal{I}_t(\mathsf{n})$

$$\mathcal{U}_t^D(\mathsf{x}; i^t) = U^D \left(\mathsf{x}_t(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)} \int_0^{\mathsf{n}_{t+1}(i^t)} \mathcal{U}_{t+1}^D(\mathsf{x}; i^t, i_{t+1}) di_{t+1} \right),$$

where $U^D : \mathcal{X} \times \mathbb{R}^* \longrightarrow \mathbb{R}^*$ is a non-decreasing function. Thus, parents' preferences on their children's decisions might differ from the preferences of these children.

To conclude the definitions, for every allocation $\mathsf{a} \in \mathcal{X} \times \mathcal{K}$ and every $t \geq 0$ and $i^t \in \mathcal{I}_t(n)$, write $\mathsf{e}_t(i^t)$ for the amount of physical resources –or *income*– available for agent i^t at period t; that is,

$$\mathsf{e}_t(i^t) := c_t^m(i^t) + b_t(\mathsf{n}_{t+1}(i^t)) + k_{t+1}^o(i^t).$$

Consider an arbitrary i^t and e_t, and let $\mathcal{F}(e_t; i^t)$ be the set formed by all sequences $\{(\mathsf{x}_\tau, \mathsf{k}_{\tau+1}^o) : \mathbb{R}_+^t \longrightarrow \mathbb{R}_+^4\}_{\tau \geqslant t+1}$ satisfying $\mathsf{e}_t(i^t) \leq e_t$ and, for all $\tau \geqslant t+1$, the feasibility constraint that agent i^t's descendants would face at τ if they were not allowed to obtain resources from other agents in the economy, that is

$$\int_{\mathcal{D}_{\tau-1}(i^t)} \mathsf{c}_\tau^o(i) di + \int_{\mathcal{D}_\tau(i^t)} \left[\mathsf{c}_\tau^m(i) + b_\tau(\mathsf{n}_{\tau+1}(i)) + \mathsf{k}_{\tau+1}^o(i) \right] di \leq \int_{\mathcal{D}_{\tau-1}(i^t)} F_\tau(\mathsf{k}_\tau^o(i), \mathsf{n}_\tau(i)) di.$$

For each $t \geq 1$ and $e_t \geq 0$, let $\mathcal{V}_t^D(e_t)$ be defined as the maximal utility that the dynasty head can obtain from their descendants born at t by endowing any of their immediate descendants with e_t units of resources, that is[2]

$$\mathcal{V}_t^D(e_t) := \max_{i^t \in \mathbb{R}^t} \left\{ \max_{(\mathsf{x}, \mathsf{k}^o) \in \mathcal{F}(e_t; i^t)} \mathcal{U}_t^D(\mathsf{x}; i^t) \right\}. \tag{4}$$

With this notation, the maximum utility that the dynasty head can obtain with a feasible allocation can be written as

$$\mathcal{V}_0(\bar{e}_0) = \max_{\substack{(x_0, k_1^o) \in \mathbb{R}_+^4 \\ \mathsf{e} : \mathbb{R}_+ \longrightarrow \mathbb{R}_+}} \left\{ U \left(x_0, \frac{1}{n_1} \int_0^{n_1} \mathcal{V}_1^D(\mathsf{e}(i)) di \right) : c_0^m + b_0(n_1) + k_1^o \leq \bar{e}_0;\ c_1^o + \int_0^{n_1} \mathsf{e}(i) di \leq F_1(k_1^o, n_1) \right\}. \tag{5}$$

We assume that for each $t \geq 1$, the function $\mathcal{V}_t^D$ is well defined on $\mathbb{R}_+$, and that $\mathcal{V}_0(\bar{e}_0) < \infty$ is satisfied.

Throughout the paper, we shall impose the following additional assumptions on preferences.

*A*1 The functions $U^D(\cdot)$ and $U(\cdot)$ are non-decreasing, concave and continuously differentiable on $\mathbb{R}_{++}^4$, and the function $U(\cdot)$ is strictly increasing in c_t^m and n_{t+1}.

[2]Since the utility received by the dynasty head from consumption of any of her descendants is the same, any choice of i^{t+1} in the optimization problem in the definition of $\mathcal{V}_t^D(e_t)$ is optimal.

$A2$ For any fixed $u^D \in \mathbb{R}^*$ and any two $(x, \widetilde{x}) \in \mathbb{R}^3_+ \times \mathbb{R}^3_+$, $U^D\left(x, u^D\right) \geq U^D\left(\widetilde{x}, u^D\right)$ is satisfied whenever $U\left(x, u^D\right) \geq U\left(\widetilde{x}, \widetilde{u}^D\right)$ is satisfied.

$A3$ For any two $\left(x, u^D\right) \in \mathbb{R}^3_+ \times \mathbb{R}^*$ and $\left(\widetilde{x}, \widetilde{u}^D\right) \in \mathbb{R}^3_+ \times \mathbb{R}^*, U\left(x, u^D\right) > U\left(\widetilde{x}, \widetilde{u}^D\right)$ is satisfied whenever $U^D\left(x, u^D\right) > U^D\left(\widetilde{x}, \widetilde{u}^D\right)$ is satisfied.

Assumptions $A2$ and $A3$ are placed to characterize Millian efficient allocations, dynastic optima and equilibrium allocations in terms of indirect utility functions (defined formally in Sections 3 and 6) that depend exclusively on the amount of resources available to finance consumption, fertility and investment decisions taken by the agents. Taken together, these two assumptions mean that the agents preferences are, in a sense that we shall make clearer below, consistent. To be more precise, $A2$ implies that, keeping fixed the total amount of resources available to any given agent and the decisions taken by the agent's descendants, the agent's preferences on how to distribute these resources among consumption, fertility and investment coincide with those of her parents; while $A3$ implies that, whenever an agent is willing to increase the total resources available to any of her grandchildren –and, hence, to increase the utility that the agent obtains from consumption decisions of her grandchildren–, then the agent's children agree on that decision. Put it in other words, the agents discount the utility obtained by their grandchildren at least at the same rate as their parents do.

Symmetric allocations. Observe that preferences and labor capacities of any two agents of the same generation are identical –i.e., if any two alive agents of any generation (and all their descendants) take the same decisions–, then they obtain the same welfare. In view of this, it seems innocuous, both from normative and positive concerns, to restrict attention to *symmetric* allocations, that is, to allocations for which any two agents of the same generation choose the same consumption and fertility bundles. Formally, a feasible allocation $\mathsf{a} \in \mathcal{F}$ is said to be *symmetric* if for any t and any two agents i^t, $\tilde{i}^t \in \mathcal{I}_t(\mathsf{n})$ one has $\mathsf{x}_t(i^t) = \mathsf{x}_t(\tilde{i}^t) = x_t$ and $\mathsf{k}^o_{t+1}(i^t) = \mathsf{k}^o_{t+1}(\tilde{i}^t) = k^o_{t+1}$. A symmetric allocation is thus represented by a pair of sequences $(x, k^o) \in \mathcal{X}^S \times \mathcal{K}^S$, where $\mathcal{X}^S$ is the set of all sequences $x = \{(x_t)\}_{t=0}^{\infty}$ of non-negative vectors $x_t = (c^m_t, c^o_{t+1}, n_{t+1}) \in \mathbb{R}^3_+$ and $\mathcal{K}^S$ is the set of all sequences $k^o = \left\{\left(k^o_{t+1}\right)\right\}_{t=0}^{\infty}$ of non-negative real numbers. Within symmetric allocations, for $t \geq 0$, the resource constraint in (2) reduces to

$$c^o_t + n_{t+1}\left[c^m_t + b_t(n_{t+1}) + k^o_{t+1}\right] \leq F_t(k^o_t, n_t), \tag{6}$$

where $k^o_{t+1} = K_{t+1}/N_t$ represents capital invested *per* old adult. A symmetric allocation (x, k^o) is feasible if it satisfies, for each $t \geq 0$, the resource constraint in (6) and the initial condition $c^m_0 + b_0(n_1) + k^o_1 \leq \bar{e}_0$. Denote by $\mathcal{S} \subset \mathcal{X}^S \times \mathcal{K}^S$ the set containing all feasible symmetric allocations.

Note that for every t, the welfare obtained by the dynasty head from consumption and fertility decisions of every two alive agents i^t and $\tilde{i}^t$ of generation t with an allocation $\mathsf{a} \in \mathcal{S}$ satisfies $\mathcal{U}^D_t(\mathsf{x}; i^t) = \mathcal{U}^D_t(\mathsf{x}; \tilde{i}^t) = U^D_t(x)$, where $U^D_t : \mathcal{X}^S \longrightarrow \mathbb{R}$ is recursively defined, for each t, by $U^D_t(x) = U^D\left(x_t, U^D_{t+1}(x)\right)$. Thus, the welfare obtained by an agent of generation t with a symmetric allocation is

$$U_t(x) = U(x_t, U^D_{t+1}(x)).$$

On types of altruism. *Dynastic altruism.* To understand the role of our assumptions, it is useful to take, as a benchmark, the case in which $U^D \equiv U$ and U is strictly increasing in c_t^m, n_{t+1} and u_{t+1}^D. In this particular environment, every agent cares about the utility of their immediate descendants, which, proceeding recursively, implies that every agent cares about consumption and fertility decisions of all her descendants. This type of altruism, first introduced by Barro (1974) in a setting with exogenous fertility, is usually referred to as *dynastic*, *recursive* or *perfect*. In this case, assumptions $A2$ and $A3$ are trivially satisfied, and $A1$ is standard and ensures that the indirect utility functions associated to the various notions of efficiency are well defined. Observe that, with Assumption $A1$, we are not imposing that $U(\cdot)$ is strictly monotonic in c_{t+1}^o, which allows us to consider, as different specifications of dynastic altruism, *i*) models in which the agents live for one period and provide with bequests to their immediate descendants, as well as *ii*) models in which the different generations of agents are truly overlapping and the agents provide their immediate descendants with gifts. Examples of the first type of models –although they restrict their analysis to symmetric allocations– are the pioneering work by Razin and Ben-Zion (1975), for whom

$$U\left(x_t, u_{t+1}^D\right) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \gamma(n_{t+1}) + \beta u_{t+1}^D;$$

as well as the model developed in Barro and Becker (1989), for whom

$$U\left(x_t, u_{t+1}^D\right) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \alpha(n_{t+1}) n_{t+1} u_{t+1}^D.$$

with $\alpha(n_{t+1}) = \alpha n_{t+1}^{-\epsilon}$ is the endogenous discounting rate. Finally, an example of a model with dynastic altruism and truly overlapping generations is Schoonbroodt and Tertilt (2014), for whom

$$U\left(x_t, u_{t+1}^D\right) = U^D(x_t, u_{t+1}^D) = v(c_t^m) + \beta v(c_{t+1}^o) + \Psi(n_{t+1}, u_{t+1}^D).$$

No altruism. In many other models studying fertility, the agents are not altruistic at all, and children are viewed as a consumption good.[3] Since we are not imposing that U or U^D must be strictly monotonic in u_{t+1}^D, a setting with no altruism is a particular specification of our general framework, for which

$$U\left(x_t, u_{t+1}^D\right) = u(x_t) \text{ and } U^D\left(x_t, u_{t+1}^D\right) = \overline{u}.$$

In this case, Assumptions $A2$ and $A3$ are trivially satisfied.

Non-dynastic altruism. But there exist other possibilities. In the literature of exogenous fertility, some authors[4] have studied environments with *limited*, or *non-dynastic* altruism to study to what extent the positive (for example, *Ricardian Equivalence*) or normative (efficiency) properties of the equilibria arising with dynastic altruism can be extended to more general settings. The literature of endogenous fertility is also abundant on specifications of altruism in which the *quality* of children, from which parents derive utility, is not necessarily

[3]Examples of this approach –focusing exclusively on symmetric allocations– are Eckstein and Wolpin (1985), Michel and Wigniolle (2007) or Conde-Ruiz *et al.* (2010).

[4]See, e.g., Bernheim and Ray (1989) and the references therein.

identified with children's utilities, and may take the form of goods spent on each child, as in Becker and Lewis (1973); income, as in Galor and Weil (2000); human capital, as in De la Croix and Doepke (2005); or consumption, as in Kollmann (1997).[5]

A particular specification of non-dynastic altruism is that for which an agent's altruism extends towards all her future descendants, which corresponds, for example, to the case in which $U^D(x_t, u^D_{t+1}) = U(x_t, \beta u^D_{t+1})$, with $0 < \beta < 1$. However, the function U^D needs not be strictly increasing in u^D_{t+1}, that is, the agents might be altruistic only towards their immediate descendants. We shall refer to this type of non-dynastic altruism as *finite horizon,* non-dynastic altruism, which is captured by utility functions of the form

$$U\left(x_t, u^D_{t+1}\right) = v(x^m_t) + \delta u^D_{t+1} \text{ and } U^D(x_t, u^D_{t+1}) = v(c^m_t).$$

3 Notions of efficiency with endogenous fertility

$\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency. According to GJT, there are at least two possible extensions of the Pareto criterion, applicable to rank allocations with different fertility choices. The first of these extensions, referred to as the $\mathcal{A}-$dominance criterion, ranks any two allocations by applying the Pareto criterion using information of the preference profiles of those agents who are born in the two allocations. Thus, an allocation a $\mathcal{A}-$dominates an allocation a' if for every t and every $i^t \in \mathcal{I}_t(\mathsf{n}) \cap \mathcal{I}_t(\mathsf{n}')$ one has $\mathcal{U}_t(\mathsf{x}; i^t) \geq \mathcal{U}_t(\mathsf{x}'; i^t)$, and there exists a period τ and a set $B^\tau \in \mathcal{I}_t(\mathsf{n}) \cap \mathcal{I}_t(\mathsf{n}')$ of positive measure for which $\mathcal{U}_\tau(\mathsf{x}; i^\tau) > \mathcal{U}_\tau(\mathsf{x}'; i^\tau)$ for all $i^\tau \in B^\tau$.

The second extension, the $\mathcal{P}-$dominance criterion, is constructed from a preliminary assumption on the utility level obtained by non-born agents. Formally, let $\mathcal{U}^N$ be a sequence of functions $\mathcal{U}^N \equiv \left\{\mathcal{U}^N_t : \mathcal{X} \times \mathbb{R}^t_+ \longrightarrow \mathbb{R}^*\right\}$ such that each utility function $\mathcal{U}^N_t$ in the sequence assigns, for every consumption-fertility path $\mathsf{x} \in \mathcal{X}$ and every $i^t \in \mathbb{R}^t$, an extended real number $\mathcal{U}^N_t(\mathsf{x}; i^t) \in \mathbb{R}^*$ that captures normative principles determining under what circumstances it is worth living. Then, for any t and any potential agent of generation t, let $\mathcal{U}^P_t : \mathcal{X} \times \mathbb{R}^t \longrightarrow \mathbb{R}^*$, the utility function of any potential agent of generation t, be defined, for all $(\mathsf{x}, i^t) \in \mathcal{X} \times \mathbb{R}^t$, by

$$\mathcal{U}^P_t(\mathsf{x}; i^t) = \begin{cases} \mathcal{U}_t(\mathsf{x}; i^t), & \text{if } i^t \in \mathcal{I}_t(\mathsf{n}); \\ \mathcal{U}^N_t(\mathsf{x}; i^t), & \text{otherwise.} \end{cases}$$

The notion of $\mathcal{P}-$dominance can be defined formally as follows: an allocation a $\mathcal{P}-$dominates an allocation a' if for every t and every $i^t \in \mathbb{R}^t$ one has $\mathcal{U}^P_t(\mathsf{x}; i^t) \geq \mathcal{U}^P_t(\mathsf{x}'; i^t)$, and there exists a period τ and a set of individuals with positive measure $B^\tau \in \mathbb{R}^\tau$ for which $\mathcal{U}^P_\tau(\mathsf{x}; i^\tau) > \mathcal{U}^P_\tau(\mathsf{x}'; i^\tau)$ is satisfied for all $i^t \in B^\tau$.

These two extensions of the Pareto criterion give rise to two notions of efficiency, respectively called $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, to evaluate allocations with different population

[5] Since we are imposing, with Assumptions $A2$ and $A3$, that the agents' utilities depends exclusively on their consumption decisions -as in Kollmann´s paper- we rule out other specifications of non-dynastic altruism in which the agents are concerned on their descendants investment decisions. However, our results can be easily extended to environments with non-dynastic altruism for which $A2$ and $A3$ do not hold, as we show in Pérez-Nievas *et al.* (2016).

size. A feasible allocation is $\mathcal{A}-$efficient if it is not $\mathcal{A}-$dominated by any other feasible allocation, and a feasible allocation is $\mathcal{P}-$efficient if it is not $\mathcal{P}-$dominated by any other feasible allocation.

In their paper, GJT discuss the disadvantages and advantages of the two notions of efficiency through several results and examples, most of them obtained in a setting in which the set of fertility choices available to any agent is a discrete, bounded set. The main disadvantage of $\mathcal{P}-$efficiency is that it uses information –the preferences of unborn people– that is "inherently impossible to observe" (see GJT 2007, p.1048). The main disadvantage of $\mathcal{A}-$efficiency is that it is based on welfare comparisons of potentially different sets of agents, which might give rise to "cycles and, hence, non existence" (see GJT 2007, p.1048).[6] However, in the general setting studied in their paper, the set of $\mathcal{A}-$efficient allocations is generically non-empty (Result 3), and it is non-empty in those settings in which the allocation maximizing a weighted average of the utilities of those agents that are already alive at period $t = 0$ is uniquely defined (Result 2). In view of Results 2 and 3, the disadvantage of using $\mathcal{A}-$efficiency does not seem to be very relevant, and becomes an advantage because $\mathcal{A}-$efficiency does not use any information on preferences of the unborn. Furthermore, $\mathcal{A}-$efficient allocations are either $\mathcal{P}-$efficient or are arbitrarily close to $\mathcal{P}-$efficient allocations for every possible specification of the utilities of the unborn (Proposition 3); in particular, all allocations maximizing a weighted average of the utilities of the agents alive at period $t = 0$ are both $\mathcal{A}$ and $\mathcal{P}-$efficient (Result 2). Thus, $\mathcal{A}-$efficiency can be regarded as a *robust* type of $\mathcal{P}-$efficiency, that holds irrespectively of the utilities attributed to the unborn.

Millian efficiency. Elsewhere (see Conde-Ruiz *et al.*, 2010), we have proposed an alternative notion of efficiency, referred to as Millian efficiency (or $\mathcal{M}-$efficiency), to evaluate symmetric allocations with different population size.[7] This notion results from combining the $\mathcal{A}-$dominance criterion to compare allocations with a restriction imposing symmetry on the set of allocations that can be compared using that criterion. To be more precise,[8] a feasible symmetric allocation $a \equiv \left\{ \left(x_t, k^o_{t+1} \right) \right\}_{t=0}^{\infty} \in \mathcal{S}$ is $\mathcal{M}-$efficient if there does not exist any other feasible allocation $a' \in \mathcal{S}$ such that $U_t(x') \geq U_t(x)$ for all $t \geq 0$ and $U_\tau(x') > U_\tau(x)$ for some $\tau \geq 0$.

Observe that the formal definition of Millian efficiency is entirely analogous to that of symmetric, Pareto efficiency arising in OLG models with exogenous fertility. In standard, OLG economies,[9] the literature has distinguished between *static* (or *short-run*) efficiency, which means that an allocation cannot be improved upon by a reallocation of resources involving a finite number of generations, and *dynamic* (or *long run*) efficiency, which means full efficiency.

The characterization of Millian efficient allocations presented in Conde–Ruiz *et al.* (2010) –in economies without altruism– can be extended in the general setting described

[6]For an example showing that the binary relation induced by the $\mathcal{A}-$dominance criterion might be intransitive, see Conde-Ruiz *et al.* (2004, Ex.1).

[7]See also Michel and Wigniolle (2007).

[8]In our original formulation of the notion of Millian efficiency, all those symmetric allocations for which fertility rates are zero from some period t on are also ruled out from welfare comparisons.

[9]This distinction was first introduced by Balasko and Shell (1980).

in Section 2. In this extended characterization,[10] every Millian efficient allocation $\widehat{a} \in \mathcal{S}$ satisfies, for $t \geq 0$,

$$\begin{aligned} U_t(\widehat{x}) \quad = \quad & \max_{(x_t, k^o_{t+1}) \in \mathbb{R}^4_+} \Big\{ U\left(x_t, U^D_{t+1}(\widehat{x})\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t; \\ & F_{t+1}(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1} \Big\} \equiv W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right), \end{aligned} \tag{7}$$

which, by Assumption $A2$, yields, for $t \geq 1$

$$\begin{aligned} U^D_t(\widehat{x}) \quad = \quad & \max_{(x_t, k^o_{t+1}) \in \mathbb{R}^4_+} \Big\{ U^D\left(x_t, U^D_{t+1}(\widehat{x})\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t; \\ & F_{t+1}(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1} \Big\} \equiv W^D_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x})\right). \end{aligned}$$

That is, in a Millian efficient allocation $\widehat{a}$, consumption and fertility decisions of the agents are completely determined by the total amount of resources $\widehat{e}_t$ available to the agent, the total amount of resources $\widehat{e}_{t+1}$ available to each descendant and the average utility $U^D_{t+1}(\widehat{x})$ that the agent obtains from consumption decisions of her descendants. Given a sequence $e = \{e_t : e_0 = \overline{e}_0\}_{t \geq 0}$, for each t, write e^{-t} for the sequence $e^{-t} = \{e_\tau\}_{\tau > t}$, so that the sequence e can be written as $e = (e^t, e^{-t})$. By proceeding recursively, it is straightforward to show that, in a Millian efficient allocation $\widehat{a}$, the utility obtained by a representative agent of generation t is completely determined by the sequence $(\widehat{e}_t, \widehat{e}^{-t})$. To see this, for each $t \geq 0$ and each (e_t, e^{-t}), let $w^D_t(e_t, e^{-t})$ be recursively defined by

$$w^D_t(e_t, e^{-t}) = W^D_t\left(e_t, e_{t+1}, w^D_{t+1}(e_{t+1}, e^{-(t+1)})\right) = W^D_t\left(e_t, e_{t+1}, W^D_{t+1}\left(e_{t+1}, e_{t+2}, w^D_{t+2}(e_{t+2}, e^{-(t+2)})\right)\right) = ...$$

Then, it follows from (7) that the utility obtained by any agent of generation t in a Millian efficient allocation $\widehat{a}$ can be written as

$$U_t(\widehat{x}) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{t+1}\left(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)}\right)\right).$$

As we show in Conde-Ruiz *et al.* (2010), condition (7) characterizes every statically Millian efficient allocation in economies without altruism. In our extension to environments with altruism, an allocation $\widehat{a}$ is statically Millian efficient if, and only if, it maximizes the utility of the dynasty head among all feasible, symmetric allocations that, for each $t \geq 0$, provide each agent of generation t with at least $\widehat{e}_t$ units of resources to finance her consumption and fertility decisions. With the notation introduced above, a statically Millian efficient allocation $\widehat{a}$ can be characterized as an allocation satisfying, for $t \geq 1$,

$$\begin{aligned} U^D_t(\widehat{x}) \quad &= \quad \max\left\{w^D_t(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\} \equiv v^D_t(\widehat{e}_t, \widehat{e}^{-t}) \\ &= \quad \max\left\{W^D_t(\widehat{e}_t, e_{t+1}, v^D_t(e_{t+1}, \widehat{e}^{-(t+1)}) : e_{t+1} \geq \widehat{e}_{t+1})\right\}, \end{aligned} \tag{8}$$

for $t \geq 1$, and

$$U_0(\widehat{x}) = \max\left\{w_0(\overline{e}_0, e^{-t}) : e^{-t} \geq \widehat{e}^{-t})\right\} = \max\left\{W_0\left(\overline{e}_0, e_1, v^D_1(e_1, \widehat{e}^{-1})\right) : e_1 \geq \widehat{e}_1)\right\}. \tag{9}$$

[10] See Appendix S.A. in Pérez-Nievas *et al.* (2016).

A particular Millian efficient allocation is the allocation a^* for which the restriction $e^{-t} \geq \widehat{e}^{-t}$ is not binding, that is, the allocation that maximizes the utility of the dynasty head among feasible, symmetric allocations. In such allocation,

$$U_0(x^*) = V_0(\overline{e}_0) =: \max_{e_1 \in \mathbb{R}_+} W_0\left(\overline{e}_0, e_1, V_1^D(e_1)\right)$$

where, for each $t \geq 0$, $V_t^D : \mathbb{R}_+ \longrightarrow \mathbb{R}$ is defined, for every $e_t \in \mathbb{R}_+$, by

$$V_t^D(e_t) = \max_{e_{t+1} \in \mathbb{R}_+} W_t\left(e_t, e_{t+1}, V_{t+1}^D(e_{t+1})\right).$$

We should point out that, in the characterization given above, the fact that fertility is endogenous does not play any specific role. Therefore, conditions characterizing static efficiency in an exogenous fertility setting are almost identical to (8) and (9). The only difference is that, in an exogenous fertility problem, n_{t+1} is not a choice variable in the optimization problem in the definitions of $W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)$ and $W_t^D\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)$. Conditions (8) and (9) imply the static efficiency of an allocation $\widehat{a}$ because, for any allocation a that $\mathcal{M}-$dominates $\widehat{a}$, one must have $e_{t_1} < \widehat{e}_{t_1}$ for some $t_1 > 0$, which, in turn, implies that $e_{t_2} < \widehat{e}_{t_2}$ for some $t_2 > t_1$, and so forth.

It is in the analysis of dynamic efficiency where an important difference between the properties of Millian efficient allocations and those of Pareto efficient allocations –with exogenous fertility– arise. When fertility is endogenous, the indirect utility function $W_t\left(\cdot, U_t^D(\widehat{x})\right)$ is not, in general, quasiconcave. Due to these non-convexities, standard dynamic efficiency conditions need not be valid to identify efficient paths. Yet, it can be shown that the sufficient condition for dynamic efficiency in Conde-Ruiz *et al.* (2010, Prop.5) applies also to this general setting. Thus, a sufficient condition ensuring dynamic efficiency of a statically Millian efficient path $\widehat{a}$ is that it is satisfied

$$\lim_{T\to\infty}\left(b_T'(0) - \frac{D_2F_{T+1}(\widehat{k}_{T+1}^o, \widehat{n}_{T+1})}{D_1F_{T+1}(\widehat{k}_{T+1}^o, \widehat{n}_{T+1})}\right)\frac{\widehat{n}_{T+1}}{\widehat{e}_T} > 0. \tag{10}$$

Millian efficient allocation and dynastic optima. A final remark is in order. The sequence $\{W_t^D\}_{t\geq 1}$ of indirect utility functions that characterize Millian efficient allocations is useful to characterize unrestricted dynastic optima. To see this, given a sequence $\mathsf{e} = \left\{\mathsf{e}_{t+1} : \mathbb{R}_+^{t+1} \longrightarrow \mathbb{R}_+\right\}_{t\geq 0}$ determining total expenditures of every potential agent, for every $t \geq 0$, let $E^{\mathsf{e}}{}_{t+1} : \mathbb{R}_+ \times \mathbb{R}_+^t \longrightarrow [0,1]$ be defined, for each for every $(e, i^t) \in \mathbb{R}^{t+1}$, by

$$E_{t+1}^{\mathsf{e}}(e, i^t) = \begin{cases} \left(\frac{1}{\mu_L(\mathcal{D}_{t+1}(i^t))}\right)\int_{i\in\mathcal{D}_{t+1}(i^t):\mathsf{e}_{t+1}(i)\leq e} di, & \text{if } \int_{i\in\mathcal{D}_{t+1}(i^t)} \mathsf{e}_{t+1}(i)di > 0 \\ 1, & \text{if } \int_{i\in\mathcal{D}_{t+1}(i^t)} \mathsf{e}_{t+1}(i)di = 0. \end{cases} \tag{11}$$

Thus, $E_{t+1}^{\mathsf{e}}(e, i^t)$ determines the cumulative probability that a randomly chosen, immediate descendant of i^t spends –in consumption, fertility and investment decisions– at most e units of the homogeneous good at time $t+1$ when the sequence of expenditure functions is given by e. That is, an income scheme $\{\mathsf{e}_t\}_{t\geq 1}$ implicitly defines how income is distributed among i^t's descendants without any reference to the agent's fertility choices.

For each interval $[a,\infty) \subseteq \mathbb{R}_+$, let $\Delta[a,\infty)$ be defined as the set formed by all non-decreasing (distribution) functions $E : [a,\infty) \longrightarrow [0,1]$ satisfying $\lim_{e\to\infty} E(e) = 1$. With this notation, it is straightforward to show that the sequence of value functions $\{\mathcal{V}_t^D\}_{t\geq 1}$ defined in (4), satisfies

$$\mathcal{V}_t^D(e_t) = \max_{E:\mathbb{R}_+\longrightarrow[0,1]\in\Delta\mathbb{R}_+} \left\{ W_t^D \left(e_t, \int_{\mathbb{R}_+} edE(e), \int_{\mathbb{R}_+} \mathcal{V}_{t+1}^D(e)dE(e) \right) \right\},$$

which, by writing $\mathcal{V}_{t+1}^{Davg}(e_{t+1})$ for the maximum average utility that the dynasty head obtains by providing her immediate descendants with an average income e_{t+1}, that is,

$$\mathcal{V}_{t+1}^{Davg}(e_{t+1}) = \max_{E:\mathbb{R}_+\longrightarrow[0,1]\in\Delta\mathbb{R}_+} \left\{ \int_{\mathbb{R}_+} \mathcal{V}_{t+1}^D(e)dE(e) : \int_{\mathbb{R}_+} edE(e) = e_{t+1} \right\}, \tag{12}$$

can be written equivalently as

$$\mathcal{V}_t^D(e_t) = \left\{ \max_{e_t\geq 0} W_t^D \left(e_t, e_{t+1}, \mathcal{V}_{t+1}^{Davg}(e_{t+1}) \right) \right\}.$$

Analogously, the value function $\mathcal{V}_0$ defined in (5) satisfies

$$\mathcal{V}_0(\overline{e}_0) = \max_{E:\mathbb{R}_+\longrightarrow[0,1]\in\Delta\mathbb{R}_+} \left\{ W_0 \left(\overline{e}_0, \int_{\mathbb{R}_+} edE(e), \int_{\mathbb{R}_+} \mathcal{V}_1^D(e)dE(e) \right) \right\}. \tag{13}$$

That is, when computing the allocation maximizing the utility of the dynasty head, a key step is finding the income distributions solving the optimization problem in (12) and (13). These distributions determine the optimal fertility choices of the dynasty head.

In the following section, we use this notation to study the properties of $\mathcal{A}-$efficient and $\mathcal{P}-$efficient allocations.

4 $\mathcal{A}-$EFFICIENCY

In this section, we criticize the notion of $\mathcal{A}-$efficiency from normative considerations and present our main result. This criticism is based of the following Theorem that characterizes $\mathcal{A}-$efficient allocations as those maximizing the utility of the dynasty head. All the proofs in the paper are relegated to the Appendix.

Theorem 1 *An allocation* a *is* $\mathcal{A}-$*efficient if, and only if, it satisfies*

$$\mathcal{U}_0(\mathsf{x}; i^0) = \mathcal{V}_0(\overline{e}_0).$$

The intuition of why the notion of $\mathcal{A}-$efficiency reduces to dynastic maximization is simple. Starting from any allocation a that does not maximize the utility of the dynasty head, it is always possible to find an allocation a' with more individuals that makes all people living under both a and a' obtain more utility with a'. In such allocation a', all those who were not alive in a are forced to use their endowment to maximize their parents' utilities. Thus, these new agents can provide their parents with at least the same utility than those already living in a. Moreover, since some of the agents already living in a were not maximizing their parents' utilities, parents obtain at least the same utility with fewer resources.

Below, we explore the properties of value functions –and, hence, of $\mathcal{A}-$efficient allocations– arising in different environments.

(i) $\mathcal{A}-$efficiency in environments with finite horizon altruism. Value functions are easy to compute in environments with no altruism –i.e. $U(x, u^D) = u(x)$ is satisfied– or with finite-horizon altruism –that is, $U(x, u^D) = u(x) + \delta u^D$ and $U^D(x, u^D) = \overline{u}$ are satisfied. In the first case (*no altruism*), the economy collapses because parents do not care on the welfare of children. Formally, each function U_t^D is constant and therefore $\mathcal{V}_t^D(e_t) = V_t^D(e_t) = \overline{u}$. Since W_0 is decreasing in e_1 we have, therefore, $\mathcal{V}_0(\overline{e}_0) = W_0(\overline{e}_0, 0, \overline{u})$ and, hence, in the unique allocation maximizing the utility of the dynasty head, $\mathsf{x}_t(i^t) = 0$ must be satisfied for every $t \geq 1$ and every $i^t \in \mathbb{R}^t$.

In the second case (*finite horizon altruism*), the economy collapses one period later because parents only care on the welfare of their own children. Formally, the indirect utility function W_1^D adopts the form $W_1^D(e_1, e_2, u^D) = W_1^D(e_1, e_2, \overline{u})$. Since, for every $(e_2, \overline{u})$, the function $W_1^D(\cdot, e_2, \overline{u})$ is concave, the function $\mathcal{V}_1^D$ is also concave and satisfies $\mathcal{V}_1^D(e_1) = V_1^D(e_1) = W_1^D(e_1, 0, \overline{u})$. Therefore, this case yields a unique $\mathcal{A}-$efficient allocation satisfying, in this case, $\mathsf{x}_t(i^t) = 0$ for every $t \geq 2$ and every $i^t \in \mathbb{R}^t$. The following corollary to Theorem 1 summarizes these results.

Corollary 1 *Consider an environment in which preferences exhibit no altruism or finite-horizon altruism. In this environment, the only $\mathcal{A}-$efficient allocation a is symmetric and drives the economy to a collapse in finite time, that is, $x_t(i^t) = x_t = 0$ for every $t \geq 2$ and every $i^t \in \mathbb{R}^t$.*

Thus, when parents care only about consumption decisions of their immediate descendants, achieving $\mathcal{A}-$efficiency requires that a generation of agents must devote their entire endowment –or labor capacity– to provide with resources to their parents, which drives the economy to a collapse.

(ii) $\mathcal{A}-$efficiency in environments with infinite horizon altruism. In environments with infinite horizon altruism, that is, in environments in which U^D is strictly increasing, the following corollary follows straightforwardly from Theorem 1.

Corollary 2 *Consider an environment in which preferences exhibit infinite-horizon altruism.*
i) Suppose that, for each $t \geq 1$, each value function $\mathcal{V}_t^D$ is strictly concave. Then every $\mathcal{A}$-efficient allocation must be symmetric and satisfies

$$U_0(\widehat{x}) = \mathcal{V}_0(\overline{e}_0) = V_0(\overline{e}_0).$$

ii) Suppose there exists a period t for which the value function $\mathcal{V}_t^D$ is not concave. More specifically, let $\widehat{a}$ be any allocation maximizing the utility of the dynasty head among symmetric allocations and suppose that there exists a period $t \geq 0$ for which

$$\mathcal{V}_{t+1}^D(\widehat{e}_{t+1}) - V_{t+1}^D(\widehat{e}_{t+1}) > 0.$$

is satisfied. Then the set of symmetric, $\mathcal{A}-$efficient allocations is empty.

Corollary 2.i) shows that every allocation maximizing the utility of the dynasty head – that is, every $\mathcal{A}-$efficient allocation– must be symmetric, and hence, Millian efficient. As we shall see below, when the allocation solving $\mathcal{V}_0(\bar{e}_0)$ is symmetric, it is typically unique. In this case, there is a significant difference between the properties of $\mathcal{A}-$efficient allocations and the properties of Pareto efficient allocations arising in standard settings with exogenous fertility. There, there are many efficient allocations that Pareto dominate any inefficient allocation of resources. In contrast, when the set of $\mathcal{A}-$efficient allocations reduces to a unique, symmetric allocation, there are many allocations that, despite of being $\mathcal{A}-$inefficient, are not $\mathcal{A}-$dominated by the only $\mathcal{A}-$efficient allocation.

A sufficient condition ensuring both the strict concavity of $\mathcal{V}_t^D$ and the uniqueness of the solution of the dynastic maximization problem is that the indirect utility function W_t^D is strictly concave. However, the indirect utility function is not, in general, concave, as discussed in Section 3. Corollary 2.ii) establishes that, when the unrestricted value functions ($\mathcal{V}_t^D$) associated to dynastic optimization differs from those in which the dynasty head is restricted to select symmetric allocations (V_t^D), achieving $\mathcal{A}-$efficiency imposes that agents with the same tastes and capabilities are treated differently and obtain different consumption bundles.

Notice that, in this case, there are indeed many $\mathcal{A}-$efficient allocations. Even when the solution E^* to the optimization problem in (13) is essentially unique, any sequence of schemes e for which $E^{\mathsf{e}}(e, i^0) = E^*(e)$ might correspond to an $\mathcal{A}-$efficient allocation. In particular, since the utility that the dynasty head obtains from consumption decisions of her descendants does not depend on the specific identity of these descendants, the dynasty head can select the individuals initiating the different available growth paths agents by a random device. Non-convexities in the feasible set associated to endogenous fertility may cause that the only efficient allocations are non symmetric and –possibly– stochastic, a result also present in Prescott and Townsend (1984) or Rogerson (1988). In these works, non-convexities arise, respectively, due to incentive constraints and labor indivisibilities. As we shall see below, non convexities arising in this paper are related to a high degree of substitution –in consumption or in production– between children (or labor) and other commodities.

On concavity of value functions in a model with infinite horizon altruism.[11] Under what conditions are value functions concave? If preferences of the dynasty head adopt the form

$$U^D\left(x_t, u_{t+1}^D\right) = u(x_t) + \beta n_{t+1}^{\gamma} u_{t+1}^D,$$

where $0 < \beta < 1$ and $\gamma < 1$, then it is possible to extend to our setting the results in Álvarez (1999) or Jones and Schoonbroodt (2010),[12] who establish the concavity of the value function V_t^D –that is, the value function arising when the dynasty head is constrained to select symmetric allocations– in the context of Barro and Becker's model (that is, for $u(x_t) = (c_t^m)^\sigma$ with $\sigma \in (0,1)$). More precisely, a sufficient condition ensuring the (strict) concavity of each

[11]In Pérez-Nievas *et al.* (2016) Appendix S.B, we examine this issue and provide examples of non-concave value functions.

[12]This article provides a nice discussion of the empirical implications of different extensions of Barro and Becker's model.

value function and the uniqueness of the solution to the dynastic maximization problem is that the function $U^{Al}:\mathbb{R}^4_{++}\longmapsto\mathbb{R}$, defined, for each $X_t=(N_t,C^m_t,C^o_{t+1},N_{t+1})>>0$, by

$$U^{Al}\left(N_t,C^m_t,C^o_{t+1},N_{t+1}\right)=N_t^{\gamma}u\left(\frac{C^m_t}{N_t},\frac{C^o_{t+1}}{N_t},\frac{N_{t+1}}{N_t}\right) \tag{14}$$

is (strictly) concave on $\mathbb{R}^4_{++}$. For the particular case in which u is positive valued and homogeneous of degree $\sigma>0$ –as it is in Barro and Becker's (1989) model– a sufficient condition ensuring U^{Al} is concave is, therefore, that $1>\gamma>\sigma$ is satisfied. In view of our Theorem 1, the set of $\mathcal{A}-$efficient allocations is a singleton.

Álvarez's sufficient conditions cannot be applied when $\gamma=0$, which corresponds to models in which children are seen as any other consumption good and utility functions are time separable, as in the early work of Razin and Ben-Zion (1975). That is,

$$U^D\left(x_t,u^D_{t+1}\right)=u(x_t)+\beta u^D_{t+1}. \tag{15}$$

In absence of technical progress, each function W^D_t adopts the time invariant, separable form

$$W^D_t(e_t,e_{t+1},U^D_{t+1})=W(e_t,e_{t+1})+\beta U^D_{t+1},$$

so that a sufficient condition ensuring the (strict) concavity of the –also time invariant– value functions V^D and $\mathcal{V}^D$ is that the indirect utility function W is concave on $\mathbb{R}^2_+$.

In Conde-Ruiz *et al.* (2010, Examples 1-3), we have studied in detail the concavity properties of the function W, that are also important to provide dynamic efficiency properties of statically efficient paths in an environment without altruism. As we show there, the indirect utility function W might be quasiconvex, even when the utility function u and the aggregate production function are concave. This occurs when either *a*) children and the other consumption goods are substitutes in consumption, or *b*) labor and capital are substitutes in production. If either *a*) or *b*) occurs, the policy function associated to the constrained problem in the definition of V^D may have two steady states $e_s=\{e^1_s,e^2_s\}$ for which $V(e_s)=W(e_s,e_s)/(1-\beta)$. Whenever this occurs, it is easy to find examples showing that there exists a scalar $\lambda\in(0,1)$ satisfying,

$$\mathcal{V}^D(e_s)\geq W(e_s,\lambda e^1_s+(1-\lambda)\,e^2_s)+\lambda\beta V^D(e^1_s)+(1-\lambda)\,\beta V^D(e^2_s)>V^D(e_s), \tag{16}$$

which suffices to establish that the value function $\mathcal{V}^D$ is not concave in a neighborhood of the steady state e_s and may deliver an empty set of symmetric, $\mathcal{A}-$ efficient allocations. Inequality (16) holds for many preferences and production functions. More precisely, if

$$b=1;\ \beta=0.5;\ u(x_t)=-\frac{10}{3}\left[c^m_t c^o_{t+1}n_t\right]^{-\frac{1}{10}},\ \text{and}\ F(k^o_{t+1},n_{t+1})=\left([k^o_{t+1}]^{\frac{1}{2}}+3[n_{t+1}]^{\frac{1}{2}}\right)^2,$$

then $e^1_s=7.07$, $e^2_s=127.33$ and (16) holds for $e_s=e^2_s$ and $\lambda>0.66$.

To summarize the results in this section, we have shown that using $\mathcal{A}-$dominance as a criterion to compare any two allocations yields a notion of efficiency that might be too strong. In the following section, we show that, for many specifications of the functions determining the welfare attributed to the unborn, this does not occur for $\mathcal{P}-$dominance.

5 Millian efficiency as Robust $\mathcal{P}-$efficiency

Since the $\mathcal{P}-$dominance criterion limits the way in which new people can be brought up to the economy, the notion of $\mathcal{P}-$efficiency is, in general, weaker than the notion of $\mathcal{A}-$efficiency. For example, consider a sequence of utility functions satisfying Assumption 4.a) in GJT; that is, satisfying

$$\mathcal{U}_t^N(\mathsf{x}; i^t) = \overline{u} \text{ for every } t \geq 1 \text{ and every } (\mathsf{x}; i^t) \in \mathcal{X} \times \mathbb{R}_+^t. \tag{17}$$

With this specification of the utility attributed to the unborn, a Millian efficient allocation $\widehat{a}$ for which $U_t(\widehat{x}) \leq \overline{u}$ for all $t \geq 1$ cannot be $\mathcal{P}-$dominated by an allocation with higher population size, because achieving a $\mathcal{P}-$improvement by increasing the population requires to provide all newcomers with more resources than (or at least the same as) the resources they have in $\widehat{a}$. Analogously, there is no way to achieve a $\mathcal{P}-$improvement by decreasing the population size because all the remaining living agents (the dynasty head among them) would get lower utility than that they obtain with $\widehat{a}$. Hence, an $\mathcal{M}-$efficient allocation $\widehat{a}$ such that $U_t(\widehat{x}) \leq \overline{u}$ for all $t \geq 1$ is $\mathcal{P}-$efficient.

When the utility function attributed to the unborn adopts the particular specification in (17), the way by which the notion of $\mathcal{P}-$efficiency mitigates the problems raised in the previous Section 4 is not entirely satisfactory. First, because if it is the case that $U_t(\widehat{x}) < \overline{u}$ for all $t \geqslant 1$, we find somewhat weird accepting as optimal an allocation in which all the living agents are strictly worse-off than those who are not born (no matter how the individuals in a society come to that judgement). Second, and most important, because determining whether or not an allocation is optimal (i.e. $\mathcal{P}-$efficient) becomes heavily dependent on that judgement.

However, other specifications of $\mathcal{U}^N$ may reduce this dependency, as long as the utility attributed to the unborn depends on the utility achieved by those alive at any allocation. There are several possibilities. For instance, in order to incorporate principles that restrict the circumstances in which new people are born, we might assume that no agent would be willing to be born if being born makes her worse-off than any other living sibling, which can be represented by selecting a utility function for the unborn that adopts the *Rawlsian* form defined, for every $(\mathsf{x}, i^t) \in \mathcal{X} \times \mathcal{R}^{t+1}$, by

$$\mathcal{U}_t^N(\mathsf{x}; i^{t-1}, i_t) = \begin{cases} \inf \{\mathcal{U}_t(\mathsf{x}; i^{t-1}, i_\tau) : i_\tau < i_t\}, & \text{if } i_t \leq \mathsf{n}_t(i^{t-1}) \\ \inf \{\mathcal{U}_t(\mathsf{x}; i^{t-1}, i_\tau) : i_\tau < \mathsf{n}_t(i^{t-1})\}, & \text{if } i_t > \mathsf{n}_t(i^{t-1}) \end{cases}. \tag{18}$$

An alternative possibility arises by assuming that no agent would be willing to be born if, by being born, the agent enjoys a lower welfare level than the average welfare level enjoyed by her living siblings born before her. Such preferences for the unborn can be represented by selecting a utility function for the unborn that adopts the *Average Utilitarianism* form,

$$\mathcal{U}_t^N(\mathsf{x}; i^{t-1}, i_t) = \begin{cases} \frac{1}{i_t}\left[\int_{i_\tau \leq i_t} \mathcal{U}_t(\mathsf{x}; i^{t-1}, i_\tau) di_\tau\right], & \text{if } i_t \leq \mathsf{n}_t(i^{t-1}) \\ \frac{1}{\mathsf{n}_t(i^{t-1})}\left[\int_{i_\tau \leq \mathsf{n}_t(i^{t-1})} \mathcal{U}_t(\mathsf{x}; i^{t-1}, i_\tau) di_\tau\right], & \text{if } i_t > \mathsf{n}_t(i^{t-1}) \end{cases}.$$

In these two examples, the functions determining the utility obtained by the unborn (as a function of decisions made by their living agents) in a given allocation share a common property: the utility attributed to a particular agent if unborn is a function that depends, exclusively and symmetrically, on the utility profiles of the agent's living siblings. Formally:

Property $\boldsymbol{S}$. *For every $t \geq 1$, every $i^t \in \mathbb{R}^t_+$ and every symmetric allocation a such that $\mathsf{x}_t(i^t) = x_t$ and $\mathsf{n}_{t+1}(i^t) = n_{t+1}$ one has*

$$\mathcal{U}^N_t(\mathsf{x}; i^t) = \mathcal{U}_t(\mathsf{x}; i^t) \equiv U_t(x).$$

Theorem 2 below provides conditions under which, for any specification of the utilities attributed to the unborn satisfying Property $\boldsymbol{S}$, a symmetric, $\mathcal{P}-$efficient allocation can be characterized as a Millian efficient allocation. A new definition is needed first. Given a sequence $\widehat{e} \equiv \{\widehat{e}_t\}_{t=0}^{\infty}$, for an arbitrary $t \geq 0$ and each e_t, let the restricted value function $\mathcal{V}^D_{\widehat{e},t} : [\widehat{e}_t, +\infty) \longrightarrow \mathbb{R}$ be defined, for every $e_t \in [\widehat{e}_t, +\infty)$, by

$$\mathcal{V}^D_{\widehat{e},t}(e_t) := \max_{i^t \in \mathbb{R}^t} \left\{ \max_{(\mathsf{x},\mathsf{k}^o) \in \mathcal{F}(e_t; i^t)} \left\{ \mathcal{U}^D_t(\mathsf{x}; i^t) : \mathsf{e}_\tau(i^\tau) \geq \widehat{e}_\tau \text{ for all } \tau \geq t+1 \right\} \right\}.$$

Note that, differently from the unconstrained value function $\mathcal{V}^D_t$, the constrained value function $\mathcal{V}^D_{\widehat{e},t}$ provides the maximum utility that agents born before t can obtain from consumption decisions of their descendants, *provided* each of these descendants is endowed with at least $\widehat{e}_\tau$ units of resources.

Theorem 2 *Assume Property $\boldsymbol{S}$ holds.*
i) If $\widehat{a}$ is a symmetric, $\mathcal{P}-$efficient allocation, then $\widehat{a}$ is Millian efficient;
ii) If $\widehat{a}$ is a Millian efficient allocation and, for each t, the function $\mathcal{V}^D_{\widehat{e},t}$ is concave on $[\widehat{e}_t, +\infty)$, then $\widehat{a}$ is $\mathcal{P}-$efficient.

Observe that Theorem 2 characterizes, under certain concavity assumptions, every *symmetric* $\mathcal{P}-$efficient allocation. A characterization of *any* $\mathcal{P}-$efficient allocation seems subtler, since determining whether or not a non-symmetric allocation is $\mathcal{P}-$efficient depends on the specific functional form given to each function $\mathcal{U}^N_t$. Yet, we should point out that Theorem 2 suffices to rule out, as being $\mathcal{P}-$inefficient, any symmetric allocation that is not $\mathcal{M}-$efficient (for example, Benthamite optima).

Theorem 2.*ii*) establishes that, in regular settings in which value functions are concave on a certain range, Millian efficient allocations are $\mathcal{P}-$efficient as long as each function $\mathcal{U}^N_t$ belongs to the class of functions satisfying Property $\boldsymbol{S}$. Thus, just as an $\mathcal{A}-$efficient allocation can be described as a $\mathcal{P}-$efficient allocation for which $\mathcal{P}-$efficiency holds irrespectively of the utility attributed to the unborn, Millian efficient allocations can be described as $\mathcal{P}-$efficient allocations for which $\mathcal{P}-$efficiency holds for a wide range of specifications of the utility attributed to the unborn.

6 Equilibrium behavior

After exploring the properties of the three notions of efficiency proposed in the literature, we conclude the paper by studying the efficiency properties of a decentralized mechanism in which the agents, endowed with well-defined property rights over the commodities available in the economy, are free to trade these rights (or transfer them) to pursue their own interests. With this objective, we explore the efficiency properties of a notion of decentralized equilibrium that, as in GJT, results from the combination of the notion of competitive equilibrium and the notion of subgame perfect equilibrium of a voluntary transfer game played within families. Differently from GJT, we impose that gifts cannot be negative and that parents cannot condition their gifts and bequests on their children's behavior.

More precisely, suppose there are two markets operating at each date $t \geq 0$: a financial market, that allows agents to lend (or borrow) an arbitrary amount k^o_{t+1} of the homogeneous good in period t, and obtain (or pay back) a return equal to $R_{t+1}k^o_{t+1}$ units of the same good in period $t+1$; and, a spot job market, in which labor is exchanged against the homogeneous good at a price w_t. Since the agents' preferences may exhibit descendant altruism, each type i^t of an agent of generation t might be willing to transfer, at period $t+1$, an amount $\mathsf{g}_{t+1}(i^t, i_{t+1}) \geq 0$ of the *numerarie* to each of her immediate descendants when they reach their middle age, which we may refer to as a *bequest* or a *gift* depending on whether or not the agents live for one or two periods. By choosing such gifts, parents determine the income scheme $\mathsf{e}_{t+1}(i^t, i_{t+1}) = w_t + \mathsf{g}_{t+1}(i^t, i_{t+1}) \geq w_t$ –and, therefore, the income distribution $E^{\mathsf{e}}_{t+1}(e, i^t)$ as defined in (11)– available to their descendants.

For any allocation a, each period t and each agent $i^t \in \mathbb{R}^t$, write $\overline{\mathcal{U}}^D_{t+1}(\mathsf{x}; i^t)$ for the average utility that agent i^t obtains from consumption and fertility decisions of her descendants; that is, $\overline{\mathcal{U}}^D_{t+1}(\mathsf{x}; i^t) = \frac{1}{n(i^t)} \int_0^{n(i^t)} \mathcal{U}^D_{t+1}(\mathsf{x}; i^t, i_{t+1}) di_{t+1}$. If the agents hold correct expectations both on future prices (represented by a sequence $p^{-t} \equiv \{w_\tau, R_\tau\}_{\tau \geq t+1}$) and on their descendants' future consumption decisions (represented by a sequence of functions $\mathsf{x}^{-t} = \{\mathsf{x}_\tau : \mathbb{R}^{\tau+1} \longrightarrow \mathbb{R}_+\}_{\tau \geq t+1}$), then an agent in her middle age at time t, whose income available to finance her consumption, fertility and investment decisions is given by $\mathsf{e}_t(i^t) = w_t + \mathsf{g}_t(i^t)$, and who provides their immediate descendants with an endowment described by the function $\mathsf{e}_{t+1}(\cdot) \equiv w_t + \mathsf{g}_{t+1}(\cdot)$, will choose her consumption-fertility bundle $\mathsf{x}^*_t(i^t)$ and her savings $\mathsf{k}^*_{t+1}(i^t)$ to solve

$$\max_{(x_t, k^o_{t+1}) \in \mathbb{R}^4_+} \left\{ U\left(x_t, \overline{\mathcal{U}}^D_{t+1}(\mathsf{x}; i^t)\right) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} = \mathsf{e}_t(i^t); \right.$$
$$\left. c^o_{t+1} = R_{t+1}k^o_{t+1} - n_{t+1}\left[\int_{w_{t+1}}^{\infty} e dE^{\mathsf{e}}_{t+1}(e, i^t) - w_t\right] \right\} \equiv W_{p,t}\left(\mathsf{e}_t(i^t), \int_{w_{t+1}}^{\infty} e dE^{\mathsf{e}}_{t+1}(e, i^t), \overline{\mathcal{U}}^D_{t+1}(\mathsf{x}; i^t)\right),$$

where $W_{p,t}$ is agent i^t's indirect utility function for a given of sequence of prices p. Observe that, by Assumption $A2$, any solution $\left(x^*_t, k^*_{t+1}\right)$ to the optimization problem above in the definition of $W_{p,t}(e_t, e_{t+1}, u^D_{t+1})$ is also a solution of the optimization problem in which the objective function is replaced by $U^D\left(x_t, u^D_t\right)$. We shall denote such optimization problem by $W^D_{p,t}(e_t, e_{t+1}, u^D_{t+1})$.

To simplify things, assume that the agents play Markov strategies forming a Subgame Perfect Equilibrium (Hereafter, SPE) of the game. With Markov strategies, any two agents of the same generation who receive the same income select also the same scheme to determine the income available to their immediate descendants. That is, given the income scheme e arising as the outcome of the voluntary transfers game, each t and each $e_t \geq w_t$, there exists a (conditional) distribution function $E_{t+1}(\cdot/e_t)$ with domain $[w_{t+1}, \infty)$ such that, for any two $(i^t, \tilde{\imath}^t) \in \mathbb{R}_+^t \times \mathbb{R}_+^t$ and each $e \in [w_{t+1}, \infty)$,

$$E_{t+1}^{\mathsf{e}}(e; i^t) = E_{t+1}^{\mathsf{e}}(e; \tilde{\imath}^t) = E_{t+1}(e/e_t) \text{ whenever } \mathsf{e}_t(i^t) = \mathsf{e}_t(\tilde{\imath}^t) = e_t.$$

With this assumption, the utility payoffs obtained by any agent of generation t alive in the voluntary transfer game played within families is completely determined by the sequence of prices $p^{-t} = \{w_\tau, R_\tau\}_{\tau \geq t+1}$, the income e_t available to the agent, and the sequence of schemes $E^{-t} = \{E_{\tau+1}(\cdot/\cdot)\}_{\tau \geq t}$ determining the income available to the agent's descendants.

To be more precise, the utility payoffs obtained by an arbitrary agent i^t alive with the sequence of income schemes E can be written, for $t \geq 0$, as

$$\pi_{p,t}(e_t, E^{-t}) = W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e/e_t), \int_{w_{t+1}}^{\infty} \pi_{p,t+1}^{D}(e, E^{-(t+1)}) dE_{t+1}(e/e_t)\right);$$

where $e_0 = \overline{e}_0$ and, in turn, $\pi_{p,t}^{D}(e_t, E^{-t})$ is recursively defined, for $t \geq 1$ and $e_t \geq w_t$, by

$$\pi_{p,t}^{D}(e_t, E^{-t}) = W_{p,t}^{D}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e/e_t), \int_{w_{t+1}}^{\infty} \pi_{p,t+1}^{D}(e, E^{-(t+1)}) dE_{t+1}(e/e_t)\right).$$

To explore the properties of the equilibrium, consider what the payoffs of the game would be like under the assumption that altruism is of the dynastic type, that is, under the assumption that $U \equiv U^D$ holds. With this assumption, the (indirect) utility payoffs $\pi_{p,t}^{*}(e_t, E^{-t})$ that the dynasty head would obtain with a sequence E from consumption decisions of her descendants of generation t coincides with the utility obtained by these descendants and can be recursively written, for $t \geq 0$, as

$$\pi_{p,t}^{*}(e_t, E^{-t}) = W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e/e_t), \int_{w_{t+1}}^{\infty} \pi_{p,t+1}^{*}(e, E^{-(t+1)}) dE_{t+1}(e/e_t)\right);$$

with $e_0 = \overline{e}_0$. Thus, under the assumption that $U \equiv U^D$ holds, the dynasty head will select a sequence E^* of income schemes such that, for each $t \geq 0$ and each $e_t \geq w_t$, the sequence E^{*-t} solves

$$\mathcal{V}_{p,t}^{*}(e_t) = \max_{E^{-t}} \pi_{p,t}^{*}(e_t, E^{-t}) = \max_{E^{-t}} \left\{ W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE_{t+1}(e/e_t), \int_{w_{t+1}}^{\infty} \pi_{p,t+1}^{*}(e, E^{-(t+1)}) dE_{t+1}(e/e_t)\right) \right\},$$

so that the sequence of value functions $\{\mathcal{V}_{p,t}^{*}\}_{t \geq 0}$ satisfies, for each $t \geq 0$ and each $e_t \geq w_t$,

$$\mathcal{V}_{p,t}^{*}(e_t) = \max_{E:[w_t,\infty) \to [0,1] \in \Delta[w_t,\infty)} W_{p,t}\left(e_t, \int_{w_{t+1}}^{\infty} e dE(e), \int_{w_{t+1}}^{\infty} \mathcal{V}_{p,t+1}^{*}(e) dE(e)\right).$$

It is straightforward to show that, with dynastic altruism, the sequence E^* forms a SPE of the game and maximizes the utility of the dynasty head among all possible SPE in which the agents play Markov strategies. Using Assumptions $A2$ and $A3$, it can be shown that, even in environments in which $U \equiv U^D$ does not hold, E^* is also a SPE of the game and maximizes the utility of the dynasty head among all possible SPE in which the agents play Markov strategies. In view of this, we define a decentralized equilibrium as follows:

Definition 1 *A* ***decentralized equilibrium*** *is a feasible allocation a^*, a sequence of income schemes $\mathsf{e}^* = \left\{\mathsf{e}^*_{t+1} : \mathbb{R}^{t+1}_+ \longrightarrow \mathbb{R}_+\right\}_{t\geq 0}$ and a sequence of prices $p \equiv \{w_t, R_t\}_{t\geq 1}$ such that, for each $t \geq 0$,*
i) aggregate capital (K^*_{t+1}) and labor (L^*_{t+1}) chosen by firms maximize profits, that is, $D_1F_{t+1}(K^*_{t+1}, L^*_{t+1}) = R_{t+1}$ and $D_2F_{t+1}(K^*_{t+1}, L^*_{t+1}) = w_{t+1}$ are satisfied;
ii) given e^*, each agent $i^t \in \mathcal{I}_t(\mathsf{n}^*)$ maximizes utility; that is,

$$\mathcal{U}_t(\mathsf{x}^*, i^t) = W_{p,t}\left(\mathsf{e}^*_t(i^t), \int_{w_{t+1}}^{\infty} edE^{\mathsf{e}^*}_{t+1}(e, i^t), \overline{\mathcal{U}}^D_{t+1}(\mathsf{x}^*; i^t)\right);$$

iii) capital and labor markets clear, that is, $K^*_{t+1} = \int_{i^t\in\mathcal{I}_t(\mathsf{n})} \mathsf{k}^{o*}_{t+1}(i^t)di^t$ and $L^*_{t+1} = \mathcal{N}_{t+1}(\mathsf{n}^*) = \int_{i^t\in\mathcal{I}_t(\mathsf{n})} \mathsf{n}^*_{t+1}(i^t)di^t$ are satisfied; and,
iv) e^* is the outcome of the SPE of the voluntary transfers game that maximizes the utility of the dynasty head among all possible subgame perfect equilibria (in Markov strategies) of the game, that is, $\mathcal{U}_t(\mathsf{x}^*, i^t) = \mathcal{V}^*_{p,t}(e^*(i^t))$.

It is easy to see that, when each value function $\mathcal{V}^*_{p,t}$ is strictly concave on $[w_t, \infty)$, the allocation a^* corresponding to a decentralized equilibrium must be symmetric. We should point out that conditions ensuring the concavity of each value function $\mathcal{V}^*_{p,t}$ are analogous than those ensuring the concavity of each value function $\mathcal{V}^D_t$ in settings with infinite horizon altruism, although, in order to ensure the symmetry of equilibria, it suffices that each value function $\mathcal{V}^*_{p,t}$ is concave on the interval $[w_t, \infty)$. In particular, if the utility function U adopts the Barro and Becker form $U(x_t, u^D_{t+1}) = u(x_t) + \beta n^\gamma_{t+1} u^D_{t+1}$, a sufficient condition ensuring the symmetry of equilibria is that the function U^{Al}, as defined in (14), is concave. When preferences adopt the Razin and Ben-Zion form in (15) and the indirect utility function is separable, $W_{p,t}(e_t, e_{t+1}, u^D_{t+1}) = W_{p,t}(e_t, e_{t+1}) + \beta u^D_{t+1}$, the symmetry of a given equilibrium is ensured if $W_{p,t}$ is strictly concave on the set $\{(e_t, e_{t+1}) : (e_t, e_{t+1}) \geq (w_t, w_{t+1})\}$. Therefore, even if we allow for non-symmetric strategies, the interaction of markets and families in the framework analyzed in the paper delivers, under relatively weak conditions, a symmetric allocation in which both consumption and fertility decisions are strictly positive. Theorem 3 below shows that such symmetric equilibria are (statically) Millian efficient and, if Property ***S*** holds and the equilibrium is dynamically efficient, then it is both Millian efficient and $\mathcal{P}$-efficient.

Theorem 3 *Let $\widehat{a}$ be an allocation corresponding to a symmetric competitive equilibrium.*
i) $\widehat{a}$ is statically Millian efficient.
ii) Suppose that $\widehat{a}$ is dynamically Millian efficient, and that the sequence $\{\mathcal{U}^N_t\}_{t\geq 1}$ satisfies Property ***S****. Then $\widehat{a}$ is also $\mathcal{P}-$efficient.*

Thus, for all the altruism and utility specifications considered in this paper, a version of the First Welfare Theorem holds for $\mathcal{M}-$efficiency and, hence, for $\mathcal{P}-$efficiency. Therefore, when applied to Millian efficiency or $\mathcal{P}$-efficiency, potential markets failures are of the same nature as those affecting Pareto efficient allocations in dynamic economies with exogenous fertility: although competitive equilibria are always statically efficient, they might be inefficient (or dynamically inefficient). Observe that, for the allocation $\widehat{a}$ corresponding to the competitive equilibrium, the condition for dynamic Millian efficiency (10) reduces to

$$\lim_{T\to\infty}\left(b'_T(0) - \frac{\widehat{w}_{T+1}}{\widehat{R}_{T+1}}\right)\frac{\widehat{n}_{T+1}}{\widehat{e}_T} > 0;$$

that is, long-run wages do not exceed the capitalized costs of rearing children.

In view of our results in previous sections, this implies that the First Welfare Theorem does not hold (at least, when applied to $\mathcal{A}-$efficiency) in environments with no altruism or with non-dynastic altruism. With dynastic altruism, the First Welfare Theorem might still hold if, at equilibrium prices, the non-negativity constraint on gifts is not binding and $V^*_{p,t}(\widehat{e}_t) = V_t(\widehat{e}_t) = \mathcal{V}_t(\widehat{e}_t)$ is satisfied. Otherwise, competitive equilibria are $\mathcal{A}-$inefficient. The possibility that competitive equilibria arising with voluntary transfers are $\mathcal{A}-$inefficient when the non-negativity constraint is binding has also been shown before by Schoonbroodt and Tertilt (2014),[13] who view this possibility as a market failure —or, in the authors' words, "an instance in which Coase Theorem does not apply" (p.566), that arises because parents do not have rights on their children's labor income– and suggest to correct this market failure with (fertility dependant) pension schemes.

In our view, we should take this claim on the failure of Coase theorem (and the first Welfare Theorem), as well as the policy proposals that accompany it, with caution. First of all, the fact that the efficiency in the allocation of resources depends on the initial allocation of property rights means that the so called "Coase Theorem" does not hold... when applied to $\mathcal{A}-$efficiency. But, in a standard interpretation of Coase Theorem,[14] Pareto efficiency arises from any distribution of rights –in absence of transaction costs– because Pareto inefficiency means "unexploited gains from trade", and it is not clear to us that $\mathcal{A}-$inefficiency means the same thing. Perhaps it would if 1) the agents could really trade in a market for private contracts between parents and their children in which the latter commit to compensate the former for child expenses; and 2) true preferences of potential unborn people were such that, for any allocation of resources, any potential agent would prefer to be alive rather than not being born. But this is precisely the problem: since trade on "the right to be born" is impossible, we cannot obtain the information needed to know not only whether an allocation is efficient or not, but also to know whether or not 2) holds and, consequently, to know what *efficiency* means.

Second of all, in our setting, the symmetric equilibria with strictly positive transfers from parents to children arising in a setting with dynastic altruism cannot be distinguished, from the point of view of an outside observer, from those arising with non-dynastic altruism. However, the former may be $\mathcal{A}-$efficient, while the latter are not. Moreover, in the latter case, fertility dependant pension schemes might be ineffective as a means to achieve $\mathcal{A}-$efficiency, because, in this case, the inefficiency of markets arises because parents cannot introduce clauses in their wills that force their children not to leave any bequests to their grandchildren. However, allowing and enforcing such provisions in the agents' wills would involve considerable transaction costs, and (in case altruism extends only to a finite number of periods) would drive the economy to a collapse.

A final remark is in order. One might think that, in environments in which competitive equilibria are non symmetric, such equilibria are $\mathcal{P}-$efficient. This statement might be true if altruism is dynastic and the utility function of the unborn adopts, for example, the Rawlsian form in (18). However, such statement is not, in general true. To see why, consider

[13] This paper explores, in a setting with dynastic altruism, whether or not the First Welfare Theorem holds for the equilibria arising if parents can obtain from their children only a given fraction (possibly, zero) of their income.

[14] See Mas-Collel *et al.* 1995, p.350).

an environment with finite horizon altruism in which indirect utility functions adopt the form $W_t^D(e_t, e_{t+1}, U_{t+1}^D) = W_t(e_t, e_{t+1})$ and $W_t(e_t, e_{t+1}, U_{t+1}^D) = W_t(e_t, e_{t+1}) + \delta U_{t+1}^D$, with $0 < \delta < 1$. Recall that, in that environment, each value function $\mathcal{V}_t^D$ is concave and satisfies $\mathcal{V}_t^D(e_t) = W_t(e_t, 0)$. Assuming that the utility function of the unborn adopts, say, the Rawlsian form in (18), a $\mathcal{P}-$efficient allocation can be characterized, in this environment, as an allocation maximizing the utility of the dynasty head among symmetric and non-symmetric allocations, provided each generation of individuals must obtain a utility level above a given threshold $\overline{u}_t$. But then a $\mathcal{P}-$efficient allocation must be a symmetric allocation in which $U_t(x_t^*) = W_t(e_t^*, e_{t+1}^*) + \delta W_{t+1}(e_{t+1}^*, e_{t+2}^*)) = u_t^*$. Thus, non-symmetric competitive equilibria are neither $\mathcal{A}-$ or $\mathcal{P}-$efficient.

7 Conclusions

In this paper, we have explored the properties of the notions of $\mathcal{A}-$efficiency and $\mathcal{P}-$efficiency, proposed by Golosov, *et al.* (2007), as well as the notion of Millian efficiency (Conde-Ruiz *et al.* 2010) to evaluate allocations in a general overlapping generations setting with endogenous fertility and descendant altruism. The setting includes, as particular cases, environments with infinite horizon, dynastic altruism *à la* Barro and Becker (1988), as well as environments with finite horizon or other forms of non-dynastic altruism. In a general framework, we have shown that if we evaluate efficiency without making any judgement on whether or not it is worth living –that is, if we use the notion of $\mathcal{A}-$efficiency– an important difficulty arises: in many particular specifications of the general framework, the set of $\mathcal{A}-$efficient allocations reduces to the allocation maximizing the utility of the dynasty head. In some environments (more precisely, with finite horizon altruism) some of the agents devote their entire endowment (or labor capacity) to provide with resources to their parents, which drives the economy to a collapse.

In the paper, we have argued that these difficulties can be overcome if we incorporate principles determining under what circumstances it is worth living –that is, if we use the notion of $\mathcal{P}-$efficiency. For a wide range of functions determining the welfare attributed to the unborn, every Millian efficient allocation –that is, every symmetric allocation that is not $\mathcal{A}-$dominated by any other symmetric allocation–, is $\mathcal{P}-$efficient. Finally, we have provided a version of the First Welfare Theorem by showing that *i*) every symmetric competitive equilibrium is a –statically– Millian efficient allocation; and, that *ii*) if the utility attributed to a particular agent if unborn is a symmetric function of the utility profiles of her living siblings (i.e., Property $\boldsymbol{S}$ holds) and the equilibrium is dynamically efficient, then it is both Millian efficient and $\mathcal{P}$-efficient.

Thus, the notion of Millian efficiency (or $\mathcal{P}-$efficiency) seems more appropriate than that of $\mathcal{A}-$efficient to evaluate allocations, specially in settings in which altruism is not of the dynastic type; with the notion of Millian efficiency, an important qualitative conclusion of Golosov *et al.* prevails: in absence of non convexities, externalities, missing markets, dynamic efficiency problems, etc., the fact that fertility decisions are endogenous does not mean that markets fail to deliver efficient allocations.

There are several directions that might be worth exploring. A first direction would be to extend the results to environments in which agents are heterogeneous. Here, we should point out that the symmetry restriction underlying the Millian notion of efficiency imposes that every two agents with the same characteristics should be treated equally, but it does not mean that agents with different characteristics must be treated equally. Thus, in models in which agents are heterogeneous in their characteristics (preferences, endowments, preferences and endowments of their ancestors and finally, the agents' order of birth with respect to their siblings), the Millian notion of efficiency may be still applicable if we regard the symmetry restriction as imposing that any two agents of the same generation with the same preferences and endowments –and for whom the preferences and endowments of all their ancestors are also equal– must be treated symmetrically. As a second direction, the definitions and results displayed in this paper can be useful to compare and assess existing institutions related to intra- and inter-family relationships on efficiency grounds.

Appendix: Proofs

Proof of Theorem 1. To prove Theorem 1, observe that dynastic maximization implies $\mathcal{A}-$efficiency, as established by Result 2 in *GTJ*. To show that $\mathcal{A}-$efficiency implies dynastic maximization, let $\widehat{\mathsf{a}}$ be an $\mathcal{A}-$efficient allocation, and write $(\widehat{x}_0, \widehat{k}_1)$ and $\widehat{e}_1$ for $(\widehat{x}_0, \widehat{k}_0) = (\widehat{\mathsf{x}}_0(i^o), \widehat{\mathsf{k}}_1^o(i^o))$ and $\widehat{e}_1 = \frac{1}{\widehat{n}_1}\int_0^{\widehat{n}_1} \widehat{\mathsf{e}}_1(i)di$. Also, for each $n_1 \in \mathbb{R}_+$, write $U_1^{D\widehat{\mathsf{a}}}(n_1)$ and $e_1^{\widehat{\mathsf{a}}}(n_1)$, respectively, for

$$U_1^{D\widehat{\mathsf{a}}}(n_1) = \frac{1}{n_1}\int_0^{n_1} \mathcal{U}_1^D(\widehat{\mathsf{x}}, i)di \quad \text{and} \quad e_1^{\widehat{\mathsf{a}}}(n_1) = \frac{1}{n_1}\int_0^{n_1} \widehat{\mathsf{e}}_1(i)di.$$

With this notation, the welfare obtained by the dynasty head by choosing n_1, if all other decisions taken by the agents are those corresponding to the allocation $\widehat{\mathsf{a}}$, can be written as

$$U_0^{\widehat{\mathsf{a}}}(n_1) \quad = \quad U\left(\overline{e}_0 - b_0(n_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, n_1) - n_1 e_1^{\widehat{\mathsf{a}}}(n_1), n_1, U_1^{D\widehat{\mathsf{a}}}(n_1)\right).$$

Observe that for any two allocations a and a' differing only at a set of measure zero, a is $\mathcal{A}-$efficient if and only if a' is $\mathcal{A}-$efficient. Taking this into account, without loss of generality, suppose that $U_1^{D\widehat{\mathsf{a}}}(\cdot)$ and $e_1^{\widehat{\mathsf{a}}}(\cdot)$ are continuous (from the left) at $n_1 = \widehat{n}_1$. With this assumption the left-hand side derivative of $U_0^{\widehat{\mathsf{a}}}$ at $\widehat{n}_1$ is well defined, and it is given by

$$\frac{d^- U_0^{\widehat{\mathsf{a}}}(\widehat{n}_1)}{dn_1} = b_0'(\widehat{n}_1)\, D_1 U\left(\widehat{x}, U_1^{D\widehat{\mathsf{a}}}(\widehat{n}_1)\right) + \left[D_2 F_1(\widehat{k}_1^o, \widehat{n}_1) - e_0^{\widehat{\mathsf{a}}}(\widehat{n}_1)\right] D_2 U\left(\widehat{x}, U_1^D(\widehat{x}_1)\right) + D_3 U\left(\widehat{x}, U_1^{D\widehat{\mathsf{a}}}(\widehat{n}_1)\right).$$

Moreover, observe that, since $\widehat{\mathsf{a}}$ is $\mathcal{A}-$efficient, one must have $d^- U_0^{\widehat{\mathsf{a}}}(\widehat{n}_1)/dn_1 \geq 0$.

With this observation in mind, we now show that every $\mathcal{A}-$efficient allocation $\widehat{\mathsf{a}}$ must satisfy

$$U_1^{D\widehat{\mathsf{a}}}(\widehat{n}_1) = \frac{1}{\widehat{n}_1}\int_0^{\widehat{n}_1} \mathcal{V}_1^D(\widehat{\mathsf{e}}(i))di = \mathcal{V}_1^{Davg}(\widehat{e}_1). \tag{19}$$

To prove that (19) holds, suppose it does not. Now select a pair $(\widetilde{n}_1, \widetilde{e}_1) \geq 0$ such that $\widetilde{e}_1 < \widehat{e}_1$ and $U_1^{D\widehat{\mathsf{a}}}(\widehat{n}) = \mathcal{V}_1^D(\widetilde{e}_1)$ are satisfied. Observe that such pair must exist since (19) is not satisfied and $U_1^{D\widehat{\mathsf{a}}}(\widehat{n}) < \mathcal{V}_1^{Davg}(\widehat{e}_1)$ holds. Then, consider an allocation a constructed from $(\widetilde{n}_1, \widetilde{e}_1)$ in such a way that the following holds:

i) for $t = 0$, $\left(\mathsf{c}_0^m(i^0), \mathsf{c}_1^o(i^0), \mathsf{k}_1^o(i^0)\right) = \left(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widehat{n}_1\widehat{e}_1 - (\widetilde{n}_1 - \widehat{n}_1)\,\widetilde{e}_1, \widehat{k}_1^o\right)$;

ii) for $t \geq 1$ and $i^t \in \mathcal{I}_t(\widehat{\mathsf{n}})$, $(\mathsf{x}_t(i^t), \mathsf{k}_{t+1}^o(i^t)) = (\mathsf{x}_t(i^t), \mathsf{k}_{t+1}^o(i^t))$;

iii) for $t = 1$ and $i \in \mathcal{I}_1(\widetilde{\mathsf{n}}_1)\backslash\mathcal{I}_1(\widehat{\mathsf{n}}_1)$, $\mathcal{U}_1^D(\mathsf{x}, i) = \mathcal{V}_1^D(\mathsf{e}_1(i)) = \frac{1}{\widehat{n}_1}\int_0^{\widehat{n}_1} U_1^D(\widehat{\mathsf{x}}, i)di = \mathcal{V}_1^D(\widetilde{e}_1)$;

iv) $t > 1$ and $i \in \mathcal{I}_1(\widetilde{\mathsf{n}}_1)\backslash\mathcal{I}_1(\widehat{\mathsf{n}}_1)$, $\mathcal{U}_t^D(\mathsf{x}, i^t) = \mathcal{V}_t^D(\mathsf{e}_t(i^t))$.

Thus, for $\widetilde{n}_1 < \widehat{n}_1$, a is an allocation with fewer individuals than those living in $\widehat{\mathsf{a}}$, in which those individuals who were already living in $\widehat{\mathsf{a}}$ receive exactly the same bundle. For $\widetilde{n}_1 > \widehat{n}_1$, $\widetilde{\mathsf{a}}$ is an asymmetric allocation that splits the population into (at least) two groups at period $t = 1$: those living under $\widehat{\mathsf{a}}$ obtain the same consumption-fertility bundle as the one they obtain in $\widehat{\mathsf{a}}$, while those who were not living under $\widehat{a}$ receive an endowment $\widetilde{\mathsf{e}}_t(i^t)$ (with $\widetilde{\mathsf{e}}_t(i^t) = \widetilde{e}_1$ for $t = 1$) and take the consumption and fertility plans that maximize the welfare of the dynasty head. It is straightforward to see that such feasible allocation $\widetilde{\mathsf{a}}$ exhibiting properties i)-iv) above exists.

Write now $U^{\mathsf{a}}(\widetilde{n}_1)$ for the utility obtained by the dynasty head with the allocation a for different selections of $\widetilde{n}_1$. Note that

$$U^{\mathsf{a}}(\widetilde{n}_1) = \begin{cases} U^{\widehat{\mathsf{a}}}(\widetilde{n}_1) & \text{if } \widetilde{n}_1 \leq \widehat{n}_1, \\ U\Big(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widehat{n}_1\widehat{e}_1 - (\widetilde{n}_1 - \widehat{n}_1)\,\widetilde{e}_1, \widetilde{n}_1, \mathcal{V}_1^D(\widetilde{e}_1)\Big) & \text{if } \widetilde{n}_1 > \widehat{n}_1. \end{cases}.$$

It is straightforward to show that the allocation a has been constructed in such a way that the left-hand side derivative of U^{a} at $\widetilde{n}_1 = \widehat{n}_1$ coincides with that of $U^{\widehat{\mathsf{a}}}$. Also, the right-hand side derivative of U^{a} at $\widetilde{n}_1 = \widehat{n}_1$ is given by

$$\frac{d^+U^{\mathsf{a}}(\widehat{n}_1)}{dn_1} = \frac{d^-U^{\mathsf{a}}(\widehat{n}_1)}{dn_1} + D_2U(\widehat{x}_0, \mathcal{V}_1^D(\widetilde{e}_1))\,(\widehat{e}_1 - \widetilde{e}_1) > 0.$$

Therefore, $U^{\mathsf{a}}(\widetilde{n}_1) > U^{\widehat{\mathsf{a}}}(\widehat{n}_1)$ must be satisfied for some $\widetilde{n}_1 > \widehat{n}_1$. Thus, if we select $\widetilde{n}_1 > \widehat{n}_1$, the allocation a provides all agents already living with the allocation $\widehat{\mathsf{a}}$ with at least the same welfare than the welfare they obtain with $\widehat{\mathsf{a}}$ and provides some of these agents –the dynasty head– with strictly higher welfare, which implies that $\widehat{\mathsf{a}}$ is $\mathcal{A}-$inefficient, a contradiction that establishes that (19) is satisfied.

To complete the proof of Theorem 1, it only remains to be shown that $\mathcal{U}_0(\widehat{\mathsf{x}}; i^0) = \mathcal{V}_0(\overline{e}_0)$ must be satisfied for every $\mathcal{A}-$efficient allocation $\widehat{\mathsf{a}}$. To show $U_0(\widehat{\mathsf{x}}, i^o) = \mathcal{V}_0(\overline{e}_0)$ must hold, suppose it does not. Taking into account that $\widehat{\mathsf{a}}$ satisfies (19), observe that this implies, in turn, that $\widehat{e}_1 > e_1(\overline{e}_0)$, where $e_1(\overline{e}_0)$ is the solution to $\mathcal{V}_0(\overline{e}_0) = \max\left\{W_0\left(\overline{e}_0, e_1, \mathcal{V}_1^D(e_1)\right) : e_1 \geq 0\right\}$. Choose now $\pi \in (0,1)$ and $\widetilde{e}_1 > 0$ in such a way that $e_1(\overline{e}_0) < \widetilde{e}_1 < \widehat{e}_1$ and

$$W_0\Big(\overline{e}_0, \pi\widehat{e}_1 + (1-\pi)\,\widetilde{e}_1, \pi\mathcal{V}_1^D(\widehat{e}_1) + (1-\pi)\,\mathcal{V}_1^D(\widetilde{e}_1)\Big) > W_0\Big(\overline{e}_0, \widehat{e}_1, \mathcal{V}_1^D(\widehat{e}_1)\Big)$$

are both satisfied. Observe that the pair $(\pi, \widetilde{e}_1)$ must exist provided $e_1(\overline{e}_0) < \widetilde{e}_1 < \widehat{e}_1$ is satisfied. Taking this into account, let $\widetilde{n}_1$ be arbitrary and consider the allocation a constructed from the pair $(\widetilde{n}_1, \widetilde{e}_1)$ as explained above and satisfying $i) - iv)$. For such allocation we have

$$U^{\mathsf{a}}(\widetilde{n}_1) = U\left(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widehat{n}_1) - \widetilde{n}_1\widehat{e}_1, \widetilde{n}_1, \mathcal{V}_1^{Davg}(\widehat{e}_1)\right),$$

if $\widetilde{n}_1 \leq \widehat{n}_1$; and,

$$U^{\mathsf{a}}(\widetilde{n}_1) = U\left(\overline{e}_0 - b_0(\widetilde{n}_1) - \widehat{k}_1^o, F_1(\widehat{k}_1^o, \widetilde{n}_1) - \widehat{n}_1\widehat{e}_1 - (\widetilde{n}_1 - \widehat{n}_1)\,\widetilde{e}_1, \widetilde{n}_1, \frac{\widehat{n}_1}{\widetilde{n}_1}\mathcal{V}_1^{Davg}(\widehat{e}_1) + \left(1 - \frac{\widehat{n}_1}{\widetilde{n}_1}\right)\mathcal{V}_1^{Davg}(\widetilde{e}_1)\right)$$

if $\widetilde{n}_1 > \widehat{n}_1$. Therefore, if $\widetilde{n}_1 > \widehat{n}_1$ is chosen in such a way that $\widetilde{n}_1\pi = \widehat{n}_1$ is satisfied, then we obtain

$$U^{\mathsf{a}}(\widetilde{n}_1) = W_0\Big(\overline{e}_0, \pi\widehat{e}_1 + (1-\pi)\,\widetilde{e}_1, \pi\mathcal{V}_1^{Davg}(\widehat{e}_1) + (1-\pi)\,\mathcal{V}_1^{Davg}(\widetilde{e}_1)\Big) > W_0\Big(\overline{e}_0, \widehat{e}_1, \mathcal{V}_1^{Davg}(\widehat{e}_1)\Big) = U^{\widehat{\mathsf{a}}}(\widehat{n}_1),$$

which implies that the allocation a $\mathcal{A}-$dominates the allocation $\widehat{\mathsf{a}}$, a contradiction that establishes that $\mathcal{U}_0(\widehat{\mathsf{x}}; i^0) = \mathcal{V}_0(\overline{e}_0)$ must be satisfied and completes the proof of Theorem 1. ■

Proof of Corollary 2. Corollary 2.i) follows from (12) and Jensen's inequality, which implies that for each $t \geq 1$ and each e_t, the distribution solving $\mathcal{V}_t^{Davg}(e_t)$ must be a degenerate distribution. Corollary 2.ii) is straightforward. ■

Proof of Theorem 2. To prove i), assume Property $\boldsymbol{S}$ holds and let $\widehat{a}$ be a symmetric, $\mathcal{P}-$efficient allocation. To show that $\widehat{a}$ must be Millian efficient, suppose it is not; that is, suppose there exists an alternative symmetric allocation a that provides all generations of agents living in $\widehat{a}$ with higher utility. Since Property $\boldsymbol{S}$ holds, choosing a instead of $\widehat{a}$ implies a welfare improvement from the point of view of the $\mathcal{P}-$dominance criterion, which contradicts the assumption imposing that $\widehat{a}$ is $\mathcal{P}-$efficient and, hence, completes the proof of the i) statement in Theorem 2.

To prove ii), assume Property $\boldsymbol{S}$ holds and let $\widehat{a}$ be a Millian efficient allocation such that each function $\mathcal{V}_{\widehat{e},t}^D$ is concave on $[\widehat{e}_t, +\infty)$. To show that $\widehat{a}$ is $\mathcal{P}-$efficient, let t be arbitrary and write $\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t)$ as

$$\mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t) = \max_{E:[\widehat{e}_{t+1},\infty]\to[0,1]} W_t^D\left(\widehat{e}_t, \int dE(e), \int \mathcal{V}_{\widehat{e},t+1}^D(e)dE(e)\right),$$

which taking into account that $\mathcal{V}_{\widehat{e},t+1}^D$ is concave on $[\widehat{e}_{t+1}, +\infty)$, implies that the allocation that maximizes the utility of the dynasty head among all allocations satisfying $\mathsf{e}_{t+1}(i) \geq \widehat{e}_{t+1}$ for each $i^{t+1} \in \mathcal{I}_{t+1}(n)$ must be symmetric and, hence, it must maximize the utility of the dynasty head among all feasible symmetric allocations satisfying $e_t \geq \widehat{e}_t$ for each $t \geq 1$. Since, by Proposition S.2, the Millian efficient allocation $\widehat{a}$ satisfies this property, it follows that

$$U_t(\widehat{x}) = W_t^D(\widehat{e}_t, \widehat{e}_{t+1}, v_{t+1}^D(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)})) = \mathcal{V}_{\widehat{e},t}^D(\widehat{e}_t) \text{ for each } t \geq 1;$$

which, in turn, yields,

$$U_0(\widehat{x}) = W_0\left(\overline{e}_0, \widehat{e}_1, v_1^D(\widehat{e}_1, \widehat{e}^1)\right) = \mathcal{V}_{\widehat{e},0}^D(\overline{e}_0) \text{ for each } t \geq 1.$$

To show that this implies that $\widehat{a}$ must be $\mathcal{P}-$efficient, suppose it is not. That is, there exists an allocation a such that:
1) all agents living in both a and $\widehat{a}$ obtain at least the same welfare than the welfare they obtain with $\widehat{a}$, that is

$$\mathcal{U}_t(\mathsf{x}; i) \geq U_t(\widehat{x}) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, v_{t+1}^D(\widehat{e}_{t+1}, \widehat{e}^{-(t+1)})\right) \text{ for every } i^t \in \mathcal{I}_t(\mathsf{n}) \cap \mathcal{I}_t(\widehat{n});$$

2) all agents that are not born in $\widehat{a}$ but are born in a (that is, for which $\mathsf{n}_t(i^{t-1}, i_t) > \widehat{n}_t$) obtain, in the latter allocation, more welfare than the welfare attributed to the unborn in the allocation $\widehat{a}$. Since the welfare attributed to the unborn satisfies Property $\boldsymbol{S}$ we have $\mathcal{U}_t^N(\widehat{\mathsf{x}}; i^t) = U_t(\widehat{x})$. Therefore,

$$\mathcal{U}_t(\mathsf{x}; i^t) \geq \mathcal{U}_t^N(\widehat{\mathsf{x}}; i^t) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, v_{t+1}^D(e_{t+1}, \widehat{e}^{-(t+1)})\right) \forall i^t : i^t \in \mathcal{I}_t(\mathsf{n}) \text{ and } i \notin \mathcal{I}_t(\widehat{n}).$$

It follows that, in the allocation a, all agents must obtain a higher income than the income they obtain with $\widehat{a}$, that is, $\mathsf{e}_{t+1}(i) \geq \widehat{e}_{t+1}$ must be satisfied for each $t \geq 0$ and each $i^{t+1} \in \mathcal{I}_{t+1}(n)$. Since I) implies that $\mathcal{U}_0(\mathsf{x}; i) \geq U_0(\widehat{x})$, this contradicts the fact that $U_0(\widehat{x}) = \mathcal{V}_{\widehat{e},0}^D(\overline{e}_0)$, a contradiction that establishes that $\widehat{a}$ must be $\mathcal{P}-$efficient and completes the proof of Theorem 2. ■

Proof of Theorem 3. To prove Theorem 3, let $\widehat{a}$ be a symmetric, decentralized equilibrium associated to a sequence of prices p. By letting $w_{p,t}^*(e_t, e^{-t})$ be recursively defined, for each t, by $w_{p,t}^*(e_t, e^{-t}) = W_{p,t}(e_t, e_{t+1}, w_{p,t}^*(e_{t+1}, e^{-(t+1)}))$, it is straightforward to show that sequence $\{\widehat{e}_t\}_{t\geq 1}$ corresponding to a decentralized equilibrium solves, for each $t \geq 0$ the optimization problem

$$v_{p,t}^*(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w_{p,t}^*(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\}.$$

By Assumption $A3$, the sequence $\{\widehat{e}_t\}_{t\geq 1}$ also solves

$$v_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w_{p,t}^D(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\};$$

where $w_{p,t}^D(e_t, e^{-t})$ is recursively defined, for each t, by $w_{p,t}^D(e_t, e^{-t}) = W_{p,t}^D(e_t, e_{t+1}, w_{p,t+1}^D(e_{t+1}, e^{-(t+1)}))$. Taking this into account, we now show that one must have, for $t \geq 1$,

$$w_t^D(\widehat{e}_t, \widehat{e}^{-t}) = v_t^D(\widehat{e}_t, \widehat{e}^{-t}) = \max\left\{w_t^D(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e}^{-t}\right\}. \tag{20}$$

To prove (20) is satisfied, suppose it is not. That is, there exists $\{\widetilde{e}_t\}_{t\geq 1}$ and $\tau \geq 1$ for which $\widetilde{e}^{-\tau} \geq \widehat{e}^{-\tau}$ and $w_\tau^D(\widetilde{e}_\tau, \widetilde{e}^{-\tau}) \geq w_\tau^D(\widehat{e}_\tau, \widehat{e}^{-\tau})$ is satisfied. Also, since in a competitive equilibrium firms maximize profits one must have

$$F_{t+1}(\widehat{k}_{t+1}, \widehat{n}_{t+1}) - R_{t+1}\widehat{k}_{t+1} - w_{t+1}\widehat{n}_{t+1} = 0 \geq F_{t+1}(\widetilde{k}_{t+1}^o, \widetilde{n}_{t+1}) - R_{t+1}\widetilde{k}_{t+1}^o - w_{t+1}\widetilde{n}_{t+1}.$$

Then, $F_{t+1}(\widetilde{k}_{t+1}^o, \widetilde{n}_{t+1}) - \widetilde{n}_{t+1}\widetilde{e}_{t+1} \leq R_{t+1}\widetilde{k}_{t+1}^o + w_{t+1}\widetilde{n}_{t+1}$ and, therefore, the sequence $\{(\widehat{x}_\tau^p, \widehat{k}_\tau^p)\}_{\tau=t}^\infty$ that solves the sequence of optimization problems in the definition of $w_t^D(\widetilde{e}_\tau, \widetilde{e}^{-t})$ is feasible in the sequence of optimization problems in the definition of $w_{p,t}^D(\widetilde{e}_t, \widetilde{e}^{-t})$. This yields,

$$w_{p,t}^D(\widehat{e}_t, \widehat{e}^{-t}) = w_t^D(\widehat{e}_t, \widehat{e}^{-t}) \geq w_{p,t}^D(\widetilde{e}_t, \widetilde{e}^{-t}) \geq w_t^D(\widetilde{e}_t, \widetilde{e}^{-t}),$$

for $t \geq 1$, a contradiction that establishes (20). By Proposition S.2 in the Pérez-Nievas *et al.* (2016), the allocation $\widehat{a}$ is statically efficient.

To complete the proof, use again the profit maximizing conditions to show that a solution of the maximization problem $V_{p,t}(\widehat{e}_t)$ is also a solution of the maximization problem $\mathcal{V}_{\widehat{e},t}(\widehat{e}_t)$, hence $V_{p,t}(\widehat{e}_t) = \mathcal{V}_{\widehat{e},t}(\widehat{e}_t)$. Proceeding as in the proof of Theorem 2, this establishes that the allocation $\widehat{a}$ corresponding to a competitive equilibrium is $\mathcal{P}-$efficient, which completes the proof. ■

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Technical Appendix to "*Efficiency and Endogenous Fertility*"

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APPENDIX S.A. PROPERTIES OF MILLIAN EFFICIENT ALLOCATIONS

The properties of Millian efficient allocations arising in the general setting studied in this paper are closely analogous to those arising in the setting with no altruism that we studied in our previous work (See Conde-Ruiz *et al.*, 2010).

Proposition S.1 below shows that, in a Millian efficient allocation, consumption and fertility decisions of every alive agent are completely determined by the sequence $\widehat{e} = \{\widehat{e}_t\}_{t\geq 1}$ specifying, for each t, the amount of total expenditures $\widehat{e}_t = \widehat{c}_t^m + b_t(\widehat{n}_{t+1}) + \widehat{k}_{t+1}^o$ of each agent of generation t. Formally:

Proposition S.1 *Every $\mathcal{M}-$efficient allocation $\widehat{a} \in \mathcal{S}$ satisfies for each $t \geq 0$*

$$U_t(\widehat{x}) = \max_{(x_t, k_{t+1}^o)\in\mathbb{R}_+^4} \Big\{ U\left(x_t, U_{t+1}^D(\widehat{x})\right) : c_t^m + b_t(n_{t+1}) + k_{t+1}^o \leq \widehat{e}_t; \quad F_{t+1}(k_{t+1}^o, n_{t+1}) - c_{t+1}^o \geq n_{t+1}\widehat{e}_{t+1} \Big\} \equiv W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right), \tag{S.1}$$

which, by Assumption A2, yields, for $t \geq 1$

$$U_t^D(\widehat{x}) = \max_{(x_t, k_{t+1}^o)\in\mathbb{R}_+^4} \Big\{ U^D\left(x_t, U_{t+1}^D(\widehat{x})\right) : c_t^m + b_t(n_{t+1}) + k_{t+1}^o \leq \widehat{e}_t; \quad F_{t+1}(k_{t+1}^o, n_{t+1}) - c_{t+1}^o \geq n_{t+1}\widehat{e}_{t+1} \Big\} \equiv W_t^D\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right).$$

Proof of Proposition S.1. To prove Proposition S.1, it suffices to show that every Millian efficient allocation must satisfy $U_t(\widehat{x}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x}))$ for $t \geq 0$, which, by Assumption $A2$, implies that $U_t^D(\widehat{x}) = W_t^D(\widehat{e}_T, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x}))$ must also be satisfied for each $t \geq 1$. To show that $U_t(\widehat{x}) = W_t(\widehat{e}_\tau, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x}))$ is satisfied for $t \geq 0$, suppose that $\widehat{a}$ is an $\mathcal{A}-$efficient allocation,

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and suppose that there exists a period $\tau \geq 0$ for which the pair $(\widehat{x}_\tau, \widehat{k}^o_{\tau+1})$ corresponding to the allocation $\widehat{a}$ is not a solution to the optimization problem in the definition of $W_\tau(\widehat{e}_\tau, \widehat{e}_{\tau+1}, U^D_{\tau+1}(\widehat{x}))$. Select now a solution $(\widetilde{x}_\tau, \widetilde{k}^o_{\tau+1}) \in \mathbb{R}^4_+$ to such optimization problem and let $\widetilde{a}$ be the allocation obtained from $\widehat{a}$ by replacing the term $(\widehat{x}_\tau, \widehat{k}^o_{\tau+1})$ by such solution. This symmetric allocation is feasible because $(\widetilde{x}_\tau, \widetilde{k}^o_{\tau+1})$ must satisfy $\widetilde{c}^m_\tau + b_t(\widetilde{n}_{\tau+1}) + \widetilde{k}^o_{\tau+1} \leq \widehat{e}_\tau$ and $F_{t+1}(\widetilde{k}^o_{t+1}, \widetilde{n}_{t+1}) - \widetilde{c}^o_{t+1} \geq \widetilde{n}_{t+1}\widehat{e}_{t+1}$. Also, observe that, by assumption $A2$, the fact that $U_\tau(\widetilde{x}) > U(\widehat{x}_\tau, U^D_\tau(\widehat{x})) = U_\tau(\widehat{x})$ is satisfied, implies that $U^D_\tau(\widetilde{x}) > U^D(\widehat{x}, U^D_\tau(\widehat{x})) = U^D_\tau(\widehat{x})$ is satisfied. Therefore $U^D_t(\widetilde{x}) \geq U^D_t(\widehat{x})$ is satisfied for all $t \geq \tau$ and, since both U and U^D are monotonic, $U^D_t(\widetilde{x}) > U^D_\tau(\widehat{x})$ and $U_t(\widetilde{x}) \geq U_\tau(\widehat{x})$ are both satisfied for all $t < \tau$. That is, if the term $(\widehat{x}_\tau, \widehat{k}^o_{\tau+1})$ is not a solution to the optimization problem in the definition of $W_\tau(\widehat{e}_\tau, \widehat{e}_{\tau+1}, U^D_{\tau+1}(\widehat{x}))$, then $\widehat{a}$ is $\mathcal{M}-$dominated by an alternative allocation $\widetilde{a}$, a contradiction that establishes that both $U_t(\widehat{x}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}(\widehat{x}))$ and $U^D_{t+1}(\widehat{x}) = W^D_t(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{\tau+1}(\widehat{x}))$ must be satisfied for each $t \geq 0$ and each t. ■

Since utility and production functions are concave and differentiable, an interior solution $(\widehat{x}_t, \widehat{k}_{t+1})$ to the optimization problem in the definitions of $W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{x})\right)$ and $W^D_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U^D_t(\widehat{x})\right)$ is characterized by the two feasibility constraints

$$\widehat{c}^m_t + b_t(\widehat{n}_{t+1}) + \widehat{k}^o_{t+1} = \widehat{e}_t, \tag{S.2}$$

and

$$F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}) - \widehat{c}^o_{t+1} = \widehat{n}_{t+1}\widehat{e}_{t+1}; \tag{S.3}$$

together with the first order conditions

$$\frac{D_1U\left(\widehat{x}_t, U^D_{t+1}(\widehat{x})\right)}{D_2U\left(\widehat{x}_t, U^D_{t+1}(\widehat{x})\right)} = D_1F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}), \tag{S.4}$$

and

$$\left[b'_t(\widehat{n}_{t+1}) - \frac{D_3U\left(\widehat{x}_t, U^D_{t+1}(\widehat{x})\right)}{D_1U\left(\widehat{x}_t, U^D_{t+1}(\widehat{x})\right)}\right] D_1F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}) = D_2F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}) - \widehat{e}_{t+1}. \tag{S.5}$$

Equations (S.2) to (S.4) are almost identical to those characterizing symmetric Pareto efficient allocations in an exogenous fertility setting[1] (except for the term $b_t(\widehat{n}_{t+1})$, which in that case is assumed to be zero) and are necessary for Pareto efficiency. They simply impose feasibility and that marginal rates of substitution between current and future consumption must be equal to marginal return to investments in physical capital. The Millian notion of efficiency imposes an additional condition, represented by equation (S.5) stating that if marginal willingness to pay for children is not equal to marginal costs of rearing children, then the marginal rate of return to investments in children must be equal to the rate of return to any other investment, that is,

$$\frac{D_2F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}) - \widehat{e}_{t+1}}{b'_t(\widehat{n}_{t+1}) - \frac{D_3U\left(\widehat{x}_t, U^D_{t+1}(\widehat{x})\right)}{D_1U\left(\widehat{x}_t, U^D_{t+1}(\widehat{x})\right)}} = D_1F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1}).$$

[1]See e.g., Blanchard and Fisher (1989, p.99).

For each $t \geq 0$, write e^t for the finite sequence $e^t = (e_0, e_1, e_2, ..., e_t)$; and, write e^{-t} for the infinite sequence of non-negative real numbers $e^{-t} = \{e_\tau\}_{t \geq t+1} = (e_{t+1}, e^{-(t+1)})$. A consequence of Proposition S.1 is that, for every $t \geq 0$, the utility that any agent of generation t obtains with a Millian efficient allocation $\widehat{a}$ is entirely determined by the sequence $(\widehat{e}_t, \widehat{e^{-t}})$. More precisely,

$$U_t(\widehat{x}) = w_t(\widehat{e}_t, \widehat{e^{-t}}) \equiv W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{t+1}(\widehat{e}_{t+1}, \widehat{e^{-(t+1)}})\right),$$

where $w^D_t(\widehat{e}_t, \widehat{e^{-t}})$ is recursively defined, for any $(\widehat{e}_t, \widehat{e^{-t}})$, by

$$\begin{aligned} w^D_t(\widehat{e}_t, \widehat{e^{-t}}) &= W^D_t(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{t+1}(\widehat{e}_{t+1}, \widehat{e^{-(t+1)}})) = \\ &= W^D_t(\widehat{e}_t, \widehat{e}_{t+1}, W^D_{t+1}(\widehat{e}_{t+1}, \widehat{e}_{t+2}, w^D_{t+2}(\widehat{e}_{t+2}, \widehat{e^{-(t+2)}})) = ... \end{aligned}$$

In classical, OLG economies[2] with endogenous fertility, the literature has distinguished between *static* (or *short-run*) Pareto efficiency, which means that an allocation cannot be improved upon by a reallocation of resources involving a finite number of generations, and *dynamic* (or *long run*) efficiency, which means full efficiency. This distinction is also applicable to the notion of Millian efficiency, and we shall distinguish between static and dynamic Millian efficient allocations. Formally, a symmetric, feasible allocation $\widehat{a} \in \mathcal{S}$ is *statically* $\mathcal{M}-$efficient if there does not exist another symmetric, feasible allocation $\widetilde{a} \in \mathcal{S}$ and a finite period $T \geq 0$ such that:

i) $\widehat{a}_t = \widetilde{a}_t$ for all $t > T$;

ii) for all t such that $0 \leq t \leq T$ one has $U_t(\widetilde{x}) \geq U_t(\widehat{x})$; and,

iii) there exists t such that $0 \leq t \leq T$ and $U_t(\widetilde{x}) > U_t(\widehat{x})$ is satisfied.

In environments with no altruism, condition (S.1) in Proposition S.1 is sufficient to establish the static efficiency of an allocation satisfying this condition. In a general setting, static, Millian efficiency can be characterized as follows:

Proposition S.2

i) Every $\mathcal{M}-$efficient allocation $\widehat{a} \in S$ satisfies (S.1) and

$$\begin{aligned} U^D_t(\widehat{x}) &= \max\left\{w^D_t(\widehat{e}_t, e^{-t}) : e^{-t} \geq \widehat{e^{-t}}\right\} \equiv v^D_t(\widehat{e}_t, \widehat{e^{-t}}) \\ &= \max\left\{W^D_t(\widehat{e}_t, e_{t+1}, v^D_t(e_{t+1}, \widehat{e^{-(t+1)}}) : e_{t+1} \geq \widehat{e}_{t+1})\right\}, \end{aligned}$$

for each $t \geq 1$; and

$$\begin{aligned} U_0(\widehat{x}) &= v_0(\overline{e}_0, \widehat{e^{-0}}) \equiv \max_{aeS}\left\{U_0(x) : e^{-0} \geq \widehat{e^{-0}}\right\} \\ &= \max\left\{W_0(\overline{e}_0, e_1, v^D_1(e_1, \widehat{e^{-1}})) : e_1 \geq \widehat{e}_1\right\}, \ \textit{for } t = 0. \end{aligned}$$

[2]This distinction was introduced first by Balasko and Shell (1980).

ii) *Moreover, an allocation* $\widehat{a}$ *satisfying the necessary conditions in* i) *is statically* $\mathcal{M}-$*efficient.*

Proof of Proposition S.2. To prove that $U_t^D(\widehat{x}) = v_t^D(\widehat{e}_t, \widehat{e}^{-t})$ must be satisfied for each $t \geq 1$, write $v_t^D(\widehat{e}_t, \widehat{e}^{-t})$ as

$$v_t^D(\widehat{e}_t, \widehat{e}^{-t}) = \max_{e_{t+1} \geq 0} \left\{ W_t^D \left(\widehat{e}_t, e_{t+1}, v_{t+1}^D(e_{t+1}, \widehat{e}^{-(t+1)}) \right) \right\},$$

and suppose that there exists a period $\tau \geq 0$ for which the sequence $\widehat{e}^{-\tau}$ corresponding to the allocation $\widehat{a}$ is not a solution to the optimization problem in the definition of $v_\tau^D(\widehat{e}_\tau, \widehat{e}^{-\tau})$. Select now a sequence $\widetilde{e}^{-\tau} = \{\widetilde{e}_s\}_{s>\tau}$ such that $v_\tau^D(\widehat{e}_\tau, \widetilde{e}^{-\tau}) > v_\tau^D(\widehat{e}_\tau, \widehat{e}^{-\tau})$ is satisfied for $\tau \geq 0$, and let now $\widetilde{a}$ be the symmetric allocation for which $U_t^D(\widetilde{x}) = v_t^D(\widehat{e}_t, \widetilde{e}^{-t})$ is satisfied. Note that

$$U_t^D(\widetilde{x}) = W_t^D \left(\widehat{e}_t, \widetilde{e}_{t+1}, U_{t+1}^D(\widetilde{x}) \right) \geq W_t^D \left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x}) \right) = U_t^D(\widehat{x})$$

must be satisfied for each $t \geq 1$. Moreover, the above inequality must be satisfied as a strict inequality for $t = \tau$. Finally, Assumption *A*3 and the fact that $U_t^D(\widetilde{x}) \geq U_t^D(\widehat{x})$ is satisfied implies that

$$U_t(\widetilde{x}) = W_t \left(\widetilde{e}_t, \widetilde{e}_{t+1}, U_{t+1}^D(\widetilde{x}) \right) \geq W_t \left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x}) \right) = U_t(\widehat{x})$$

is also satisfied for each $t \geq 1$, and with strict inequality for $t = \tau$. This implies that the allocation $\widehat{a}$ is not $\mathcal{M}-$efficient, a contradiction that establishes that $U_t^D(\widehat{x}) = v_t^D(\widehat{e}_t, \widehat{e}^{-t})$ must be satisfied. Taking this into account, $U_0(\widehat{x}) = \max_{e_1 \geq 0} \left\{ W_0(\overline{e}_0, e_1, v_1^D(e_1, \widehat{e}^{-1})) \right\}$ follows straightforwardly, which completes the proof of the i) statement in Proposition S.2.

To prove ii), let $\widehat{a} \in S$ be an allocation satisfying the necessary conditions in i). To prove that $\widehat{a}$ is statically $\mathcal{M}-$efficient, we proceed by showing that if there exists an allocation $\widetilde{a}$ that $\mathcal{M}-$dominates the allocation $\widehat{a}$, then there must exist an infinite subsequence $\mathcal{T} = \{t_1, t_2, t_3, ...\}$ such that $\widetilde{e}_\tau < \widehat{e}$ for each $t \in \mathcal{T} \geq 1$. To prove this statement, observe that the fact that $U_0(\widehat{x}) = v_0(\overline{e}_0, \widehat{e}^{-0})$ and the fact that v_0 is non-increasing in e^{-0} is satisfied imply that $\widetilde{e}_{t_1} < \widehat{e}_{t_1}$ must be satisfied for some period $t_1 \geq 0$. Since $v_{t_1}^D$ is strictly increasing in e_{t_1} and non-increasing in e^{-t_1}, the fact that $\widetilde{e}^{-t_1} \geq \widehat{e}^{-t_1}$ is satisfied implies that

$$U_{t_1}^D(\widetilde{x}) < v_{t_1}^D(\widehat{e}_{t_1}, \widehat{e}^{-t_1}) = W_{t_1}^D \left(\widehat{e}_{t_1}, \widehat{e}_{t_1+1}, v_{t_1}(\widehat{e}_{t_1+1}, \widehat{e}^{-(T_1+1)}) \right) = U_{t_1}^D(\widehat{x})$$

must be satisfied, which, in turn, yields

$$U_0(\widetilde{x}) = w_0(\overline{e}_0, \widetilde{e}_1, \widetilde{e}_2, ..., \widetilde{e}_{t_1-1}, v_{t_1}^D(\widehat{e}_{t_1}, \widehat{e}^{-t_1})) < U_0(\widehat{x}),$$

which contradicts the assumption imposing that $\widetilde{a}$ $\mathcal{M}-$dominates the allocation $\widehat{a}$. Therefore, $U_0(\widetilde{x}) \geq U_0(\widehat{x})$ can be satisfied only if there exists t_2 for which $\widetilde{e}_{t_2} < \widehat{e}_{t_2}$ and $W_{t_2}^D(\widetilde{e}_{t_2}, \widetilde{e}_{t_2+1}, U_{t_2+1}^D(\widetilde{x})) > U_{t_2}(\widehat{x})$ is satisfied. By applying the argument recursively, the existence of the subsequence $\mathcal{T}$ is established. Also, since the allocation $\widehat{a}$ can only be dominated by a reallocation of resources involving a infinite sequence of periods of time, the allocation $\widehat{a}$ must be statically $\mathcal{M}-$efficient, which completes the proof of Proposition S.2. ■

The properties of statically Millian efficient allocation are closely analogous to those of symmetric, statically Pareto efficient allocations arising in models with exogenous fertility. The only difference between the characterization of statically efficient paths for these two notions of efficiency is that, in the former, the number of children n_{t+1} appears in the choice set of the optimization problem in the definition of $W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})\right)$. The necessary conditions in Proposition S.2.i) imply that, if the dynasty head (or, in settings with finite horizon altruism, any agent) is restricted to choose a symmetric allocation such that $e \geq \widehat{e}$, she cannot do better than she does with $\widehat{a}$. These necessary conditions imply the static efficiency of an allocation $\widehat{a}$ because, for any allocation a that $\mathcal{M}-$dominates $\widehat{a}$, one must have $e_{t_1} < \widehat{e}_{t_1}$ for some $t_1 > 0$, which, in turn, implies that $e_{t_2} < \widehat{e}_{t_2}$ for some $t_2 > t_1$, and so forth.

As for the notion of Pareto efficiency with exogenous fertility, the static, Millian efficiency of a given allocation $\widehat{a}$ does not preclude that all living agents can improve their welfare by reducing total resources accumulated by some generations of agents with the allocation $\widehat{a}$. In the proof of Proposition S.3 below, we show that if a statically efficient allocation $\widehat{a}$ is not fully $\mathcal{M}-$efficient (or, as it is usually found in the literature, dynamically $\mathcal{M}-$efficient) there must exist an infinite sequence $\{e_t\}_{t\geq 0}$ satisfying

$$e_t < \widehat{e}_t \text{ and } W_t(e_t, e_{t+1}, U_t^D(\widehat{x})) \geq W_t(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})), \text{ for each } t \geq 1, \tag{S.6}$$

which, by Assumption $A2$ implies in turn that the sequence $\{e_t\}_{t\geq 0}$ also satisfies

$$e_t < \widehat{e}_t \text{ and } W_t^D(e_t, e_{t+1}, U_t^D(\widehat{x})) \geq W_t^D(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})), \text{ for each } t \geq 1.$$

Thus, a sufficient condition ensuring dynamic efficiency of a statically Millian efficient path $\widehat{a}$ is that such sequence $\{e_t\}_{t\geq 0}$ does not exist.

Here, an important difference between the properties of Millian efficient allocations and Pareto efficient allocations –with exogenous fertility– arise. In the latter case, the set of feasible allocations is convex, and the concavity of utility function U implies that each indirect utility functions $W_t(\cdot, U_t^D(\widehat{x}))$ must be quasiconcave. By writing $\widehat{R}_{t+1}$ for $\widehat{R}_{t+1} = D_1 F_{t+1}(\widehat{k}^o_{t+1}, \widehat{n}_{t+1})$, it is straightforward to show that, when $U_t(\widehat{x}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x}))$ is satisfied and $W_t(\cdot, U_t^D(\widehat{x}))$ is quasiconcave, condition (S.6) can be written as

$$e_t < \widehat{e}_t \text{ and } \widehat{e}_t - e_t \geq -\frac{D_1 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})\right)}{D_2 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})\right)}\left(\widehat{e}_{t+1} - e_{t+1}\right) = \frac{\widehat{R}_{t+1}}{\widehat{n}_{t+1}}\left(\widehat{e}_{t+1} - e_{t+1}\right)$$

for each $t \geq 1$. Proceeding recursively, it is straightforward to show[3] that a sequence $\{e_t\}_{t\geq 0}$ satisfying (S.6) cannot exist –which implies that $\widehat{a}$ is dynamically efficient– if

$$\lim_{T\to\infty} \inf \left(\prod_{t=0}^{T} \frac{\widehat{e}_{t+1}\widehat{n}_{t+1}}{\widehat{e}_t \widehat{R}_{t+1}}\right) = 0 \tag{S.7}$$

is satisfied. In particular, for an allocation $\widehat{a}$ such that $\lim_{t\to\infty} \frac{\widehat{e}_{t+1}}{\widehat{e}_t} = 1$, $\lim_{t\to\infty} \widehat{n}_{t+1} = n^*$ and $\lim_{t\to\infty} \widehat{R}_{t+1} = R^*$, condition (S.7) can be written as the standard dynamic efficiency condition

$$R^* > n^*.$$

[3] See, e.g., Lemma 5.4 in Balasko and Shell (1980).

When fertility decisions are endogenous, the indirect utility function $W_t\left(\cdot, U_t^D(\widehat{x})\right)$ is not, in general, quasiconcave. Due to these non-convexities, standard dynamic efficiency conditions need not be valid to identify efficient paths. Yet, a sufficient dynamic efficiency condition can be obtained as follows.

For a given $\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right)$, define

$$\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right) = \inf_{(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})} \left\{ \frac{\widehat{e}_{t+1} - e_{t+1}}{\widehat{e}_t - e_t} : W_t\left(e_t, e_{t+1}, \widehat{u}_{t+1}^D\right) \geq W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right) \right\}.$$

Notice that, when $W_t\left(\cdot, \widehat{u}_{t+1}^D\right)$ is quasiconcave, the number $\pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right)$ corresponds to the slope of a indifference curve defined by $W_t\left(e_t, e_{t+1}, \widehat{u}_{t+1}^D\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, \widehat{u}_{t+1}^D\right)$ evaluated at $(\widehat{e}_t, \widehat{e}_{t+1})$. That is, for quasiconcave indirect utility functions we have

$$\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})) = -\frac{D_1 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)}{D_2 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)} = \frac{\widehat{R}_{t+1}}{\widehat{n}_t}.$$

However, when indirect utility functions are not quasiconcave, the number $\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x}))$ corresponds to the steepest slope (on the set $(e_t, e_{t+1}) << (\widehat{e}_t, \widehat{e}_{t+1})$) of the indifference curve defined by $W_t\left(e_t, e_{t+1}, U_t^D(\widehat{x})\right) = W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})\right)$. Therefore

$$\pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})) \leq -\frac{D_1 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})\right)}{D_2 W_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_t^D(\widehat{x})\right)} = \frac{\widehat{R}_{t+1}}{\widehat{n}_{t+1}},$$

and condition (S.6) can be written as

$$e_t < \widehat{e}_t \text{ and } \widehat{e}_t - e_t \geq \pi_t(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x}))\left(\widehat{e}_{t+1} - e_{t+1}\right) \text{ for each } t \geq 1.$$

Using this notation, in Proposition S.3 below we borrow directly from our previous work and provide a sufficient condition for dynamic efficiency that uses the sequence of implicit prices $\{(\widehat{R}_{t+1}, \widehat{w}_{t+1})\}_{t \geq 0}$ associated to a statically $\mathcal{M}-$efficient allocation $\widehat{a}$ satisfying (S.7).

Proposition S.3 *Consider a statically Millian efficient allocation $\widehat{a} \in S$ satisfying (S.7). If*

$$\lim_{T \to \infty} \inf \left(\frac{\widehat{e}_T}{\prod_{t=0}^{T} \pi_t\left(\widehat{e}_t, \widehat{e}_{t+1}, U_{t+1}^D(\widehat{x})\right)} \right) = 0 \tag{S.8}$$

is satisfied, then $\widehat{a}$ is also (dynamically) efficient. Furthermore, a sufficient condition ensuring (S.8) is that

$$\lim_{T \to \infty} \left(b_T'(0) - \frac{D_2 F_{T+1}(\widehat{k}_{T+1}^o, \widehat{n}_{T+1})}{\widehat{R}_{T+1}} \right) \frac{\widehat{n}_{T+1}}{\widehat{e}_T} > 0,$$

is satisfied.

Proof of Proposition S.3. To prove Proposition S.3, we first show that, if there exists an allocation $\widetilde{a}$ that $\mathcal{M}-$dominates the allocation $\widehat{a}$, then there must exist an allocation a that also dominates $\widehat{a}$ and satisfies, for $t \geq 1$

$$U_t(x) = U_t(\widehat{x}),\ U_t^D(x) = U_t^D(\widehat{x}), \text{ and } e_t < \widehat{e}_t. \tag{S.9}$$

To prove this statement, assume, without loss of generality, that $\widetilde{a}$ satisfies the necessary conditions for Millian efficiency in Proposition S.2. Taking this into account, an allocation a satisfying the required properties can be constructed from the allocation $\widetilde{a}$ as follows. Pick up any period $\tau \geqslant 1$ for which $U_\tau(\widetilde{x}) = W_\tau(\widetilde{e}_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{x})) > U_\tau(\widehat{x})$ is satisfied and select $e^1_\tau < \widetilde{e}^1_\tau$ in such a way that $W_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{x})) = U_\tau(\widehat{x})$ is satisfied. Notice that, by Assumption $A2$ one must have $W^D_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{x})) = U^D_\tau(\widehat{x})$. Then let x^1 be the allocation obtained from $\widetilde{a}$ by replacing the term $(\widetilde{x}_\tau, \widetilde{k}^o_\tau)$ by the solution (x^1_τ, k^{o1}_τ) to the optimization problem in the definition of $W_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{x}))$. Note that, since $\widetilde{a}$ satisfies the necessary conditions in Proposition S.2 and $w^D_{\tau-1}$ is non-increasing in e_t, we have $U^D_{\tau-1}(x^1) = W^D_{\tau-1}(\widetilde{e}_{\tau-1}, e^1_\tau, W^D_\tau(e^1_\tau, \widetilde{e}_{\tau+1}, U^D_{\tau+1}(\widetilde{x}))) \geq U^D_\tau(\widetilde{x})$. Thus, $U^D_{\tau-1}(x^1) \geq U^D_{\tau-1}(\widehat{x})$ and, hence, $U_{\tau-1}(x^1) \geq U_{\tau-1}(\widehat{x})$ must be satisfied, that is, the allocation x^1 dominates the allocation $\widetilde{a}$. Proceeding iteratively, it is straightforward to construct an allocation $\overline{a}$ satisfying the required properties for each t: $1 \leq t \leq \tau$, which taking into account that τ has been selected arbitrarily among those periods for which $U_t(\overline{x}) > U_t(\widehat{x})$ is satisfied, establishes Step 1). The remaining of the proof is exactly analogous to the proof of Propositions and in Propositions 4 and 5 in Conde-Ruiz *et al* (2010). ■

Dynamic efficiency and transversality conditions in dynastic problems. A particular Millian efficient allocation is the one maximizing the utility of the dynasty head among symmetric allocations, that is, the Millian efficient allocation $\widehat{a}$ for which

$$U^D_t(\widehat{x}) = v^D_t(\widehat{e}_t, 0) \equiv V_t(\widehat{e}_t), \text{ for } t \geq 1$$

and

$$U_0(\widehat{x}) = \max_{e_1 \geq 0} \left\{ W_0(\overline{e}_0, e_1, V^D_1(e_1)) \right\}.$$

Note that such allocation must satisfy the necessary condition for dynastic maximization

$$U^D_t(\widehat{x}) = \max_{e_{t+1} \geq 0} \left\{ W^D_t(\widehat{e}_t, e_{t+1}, w^D_{t+1}(e_{t+1}, \widehat{e}^{-(t+1)}) \right\}, \text{ for each } t \geq 1. \tag{S.10}$$

However, there exist many allocations satisfying this condition. In concave settings, the sequence $\{\widehat{e}_t\}_{t \geq 0}$ corresponding to a dynastic optimum can be identified as that satisfying the transversality condition

$$\lim_{T \to \infty} \inf \prod_{t=0}^{T} \left(\frac{\widehat{e}_{t+1} \widehat{n}_{t+1}}{\widehat{e}_t \widehat{R}_{t+1}} \right) = 0. \tag{S.11}$$

Unfortunately, standard transversality conditions might not work in non-convex settings. Yet, it is straightforward to show that the dynamic efficiency conditions in Proposition S.3 are sufficient transversality conditions ensuring that an allocation satisfying (S.10) corresponds to the dynastic optimum.

Other environments with finite horizon altruism. Some of the results obtained above can be easily extended to other environments in which the agents care about some of the decisions of their immediate descendants, even though Assumptions $A2$ and

$A3$ are not satisfied. As an example, suppose the agents care about their immediate descendants' wealth during their second period of life, that is, preferences are represented by utility functions of the form

$$\mathcal{U}_t(\mathsf{x}, i^t) = U\left(\mathsf{x}(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)}\int_0^{\mathsf{n}_{t+1}(i^t)} u^D(\mathsf{e}_{t+1}(i^t, i_{t+1}))di_{t+1}\right);$$

where U is strictly increasing and concave on $\mathbb{R}^4_+$, and u^D is also strictly increasing and concave. Note that, with these preferences, Assumption $A3$ is not satisfied. In this case, it is easy to prove that $U_t(\widehat{x}) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D_t(\widehat{e}_{t+1}))$ must hold for any Millian efficient allocation. Differently from the characterization given in Proposition S.3, statically efficient, Millian efficient allocations can be characterized, in this case, as those satisfying, for each $t \geq 0$

$$U_t(\widehat{x}) = \max\left\{W_t(\widehat{e}_t, e_{t+1}, u^D_t(e_{t+1})) : e_{t+1} \geq \widehat{e}_{t+1}\right\}.$$

Dynamic efficiency conditions, however, are entirely analogous to those in Proposition S.3.

In the setting described, the main results obtained in the paper must be slightly modified. First, the set of $\mathcal{A}-$efficient allocations can be characterized as those Millian efficient allocations for which, at each point in time, $\widehat{e}_{t+1}$ solves, without constraints, the optimization problem $\max\left\{W_t(\widehat{e}_t, e_{t+1}, u^D_t(e_{t+1})) : e_{t+1} \geq 0\right\}$. Thus, although $\mathcal{A}-$efficiency does not reduce to dynastic maximization and does not drive the economy to a collapse, the set of $\mathcal{A}-$efficient allocations does reduce, typically, to a singleton. Second of all, when the functions determining the welfare attributed to the unborn satisfy Property $\boldsymbol{S}$, every symmetric, $\mathcal{P}-$efficient allocation can be characterized as a Millian efficient allocation. Third, competitive equilibria –which in this setting are necessarily symmetric– are statically efficient and, under certain conditions, dynamically efficient and $\mathcal{P}-$efficient. A competitive equilibrium might also be $\mathcal{A}-$efficient, but only if the non-negativity constraint on gifts and bequests is not binding.

As another example, suppose that the agents care about their immediate descendants' middle-aged consumption during their second period of life, that is, preferences are represented by utility functions of the form

$$\mathcal{U}_t(\mathsf{x}, i^t) = U\left(\mathsf{x}(i^t), \frac{1}{\mathsf{n}_{t+1}(i^t)}\int_0^{\mathsf{n}_{t+1}(i^t)} u^D(\mathsf{c}^m_{t+1}(i^t, i_{t+1}))di_{t+1}\right);$$

where U is strictly increasing and concave on $\mathbb{R}^4_+$ and u^D is also strictly increasing and concave. In this case, it is straightforward to show that, in any $\mathcal{M}-$efficient allocation $\widehat{a}$ and for every t, the vector $a_t = (c^m_t, c^o_{t+1}, n_{t+1}, k^o_{t+1})$ must solve a problem closely analogous to that in the definition of $W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D(\widehat{c}^m_{t+1}))$, with an additional constraint of the form $c^m_{t+1} \geq \widehat{c}^m_{t+1}$. Thus, those allocations for which $U_t(\widehat{x}_t) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D_t(\widehat{c}^m_{t+1})$ –that is, those allocations for which the constraint $c^m_t \geq \widehat{c}^m_t$ is not binding– trivially satisfy such necessary condition of $\mathcal{M}-$efficiency and, therefore, form a subclass of all possible $\mathcal{M}-$efficient allocations. By writing $C_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D(\widehat{c}^m_{t+1}))$ for the solution to the optimization problem in the definition of $W_t(\widehat{e}_t, \widehat{e}_{t+1}, u^D_t(\widehat{c}^m_{t+1}))$ and by letting $w^D_t(e_t, e^{-t})$ be recursively defined by $w^D_t(\widehat{e}_t, \widehat{e}^{-t}) = u^D(C_t(\widehat{e}_t, \widehat{e}_{t+1}, w^D_{t+1}(\widehat{e}_{t+2}, \widehat{e}^{-(t+2)}))$, it is straightforward to show condition $i)$

in Proposition S.2 is necessary and sufficient to characterize statically $\mathcal{M}-$efficient paths within the class of allocations for which $U_t(\widehat{x}_t) = W_t(\widehat{e}_t, \widehat{e}_{t+1}, u_t^D(\widehat{c}_{t+1}^m))$ is satisfied. As for the previous example, dynamic efficiency conditions, however, are entirely analogous to those in Proposition S.3.

In such a setting, no matter whether the constraints $c_t^m \geq \widehat{c}_t^m$ are binding or not, the qualitative results obtained in this paper extend to environments with this type of altruism. In particular, $\mathcal{A}-$efficiency reduces to dynastic maximization, which implies that $\widehat{e}_2 = 0$ and $\widehat{c}_{t+2}^m = 0$ must be satisfied and, therefore, drives the economy to a collapse. Also, similar results to those in Theorems 2 and 3 in the main paper hold.

Appendix S.B: On concavity of value functions in a model with infinite horizon altruism

Only a few papers have explicitly studied whether or not value functions arising in standard models of endogenous fertility are concave. Two important exceptions are Álvarez (1999) and Qi and Kadaya (2010), although they both focus on value functions arising when the dynasty head is restricted to select symmetric allocations. The first paper focuses on Barro and Becker's (1989) model and shows that the optimization problem in the definition of the value function can be transformed in such a way that the feasible set is convex and the utility function is concave, which suffices to ensure concavity of the value function. The latter provides conditions ensuring the concavity of symmetric value functions in a more general setting, but it does it at a cost: fertility choices must be bounded from below by a strictly positive number.

S.B.1 Concavity of value functions in an extension of Barro and Becker's model

Although Alvarez's paper focuses on the restricted value functions arising in Barro and Becker's model, its arguments can be easily extended to the unrestricted value functions arising in more general models with truly overlapping generations. That is, to models for which costs of rearing children are linear –i.e., $b_t(n_{t+1}) = bn_{t+1}$– and that utility functions adopt the form

$$U^D\left(x_t, u^D_{t+1}\right) = u(x_t) + \beta n^\gamma_{t+1} u^D_{t+1} = u(x_t) + \beta n^\gamma_{t+1} U^D\left(x_{t+1}, u^D_{t+2}\right) = ...;$$

where $u(\cdot)$ is concave and, since we are imposing that U^D must be non decreasing, $0 \leq \gamma < 1$ is satisfied whenever $u(\cdot)$ is positive valued and $\gamma \leq 0$ is satisfied whenever $u(\cdot)$ is negative valued.

In this context, the utility $\mathcal{U}^D_t(x; i^t)$ that the dynasty head obtains from consumption and fertility decisions of her descendants of the dynasty initiated by agent i^t at t can be written as

$$\mathcal{U}^D_t(\mathsf{x}; i^t) = \sum_{\tau=t}^{\infty} \beta^{\tau-t} \int_{\mathcal{D}_\tau(i^t)} \mathsf{N}_\tau(i^\tau)^\gamma u(\mathsf{x}_\tau(i^\tau)) di^\tau,$$

where $\mathsf{N}_t(i^t) = 1$; $\int_{\mathcal{D}_t(i^t)} \mathsf{N}_t(i^t)^\gamma u(\mathsf{x}_t(i^t)) di^t = u(\mathsf{x}_\tau(i^\tau))$ and, for each $\tau > t$ and each $i^\tau \in \mathbb{R}^t_+, \mathsf{N}_\tau(i^\tau) = \mathsf{N}_\tau(i^{\tau-1}, i_\tau)$ is recursively defined by

$$\mathsf{N}_\tau(i^{\tau-1}, i_\tau) = \mathsf{N}_{\tau-1}(i^{\tau-1})\mathsf{n}_\tau(i^{\tau-1}) = \mathsf{N}_{\tau-2}(i^{\tau-2})\mathsf{n}_{\tau-2}(i^{\tau-2})\mathsf{n}_{\tau-1}(i^{\tau-1}) = \prod_{s=t+1}^{\tau} \mathsf{n}_s(i^{s-1})$$

Recall that $\mathcal{V}^D_1(e_t)$ has been defined as the maximum of $\mathcal{U}^D_t(x; i^t)$ (for an arbitrary i^t) among all sequences $\left\{(\mathsf{x}_\tau, \mathsf{k}^o_{\tau+1}) : \mathcal{D}_\tau(i^t) \longrightarrow \mathbb{R}^4_+\right\}_{\tau \geq t}$ satisfying the feasibility constraints

$$\mathsf{c}^m_t(i^t) + b\mathsf{n}_{t+1}(i^t) + \mathsf{k}^o_{t+1}(i^t) = e_t;$$

for $\tau = t$; and

$$\int_{\mathcal{D}_\tau(i^t)} \left(\mathsf{c}^o_t(i^{\tau-1}) + \mathsf{n}_\tau(i^{\tau-1})\left[\mathsf{c}^m_\tau(i^\tau) + b\mathsf{n}_{\tau+1}(i^\tau) + \mathsf{k}^o_{\tau+1}(i^\tau)\right]\right) di^\tau \leq \int_{\mathcal{D}_\tau(i^t)} F_\tau(\mathsf{k}^o_\tau(i^{\tau-1}), \mathsf{n}_\tau(i^{\tau-1})) di^\tau,$$

for $\tau > t$. By writing, for each $\tau > t$ and each $i^\tau \in \mathcal{D}_\tau(i^t)$, $\mathsf{X}_\tau(i^\tau)$ for $\mathsf{X}_\tau(i^\tau) = \mathsf{N}_\tau(i^\tau)\mathsf{x}_\tau(i^\tau)$ and $\mathsf{K}_{\tau+1}(i^\tau)$ for $\mathsf{K}_{\tau+1}(i^\tau) = \mathsf{N}_\tau(i^\tau)\mathsf{k}^o_{\tau+1}(i^\tau)$. With this notation, the optimization problem in the definition of $\mathcal{V}_1^D(e_t)$ is equivalent to that of choosing $\left\{(\mathsf{X}_\tau, \mathsf{K}^o_{\tau+1}) : \mathcal{D}_\tau(i^t) \longrightarrow \mathbb{R}^4_+\right\}_{\tau\geq t}$ to maximize

$$\begin{aligned}\mathcal{U}_t^D(\mathsf{x}; i^t) &= \sum_{\tau=t}^{\infty} \beta^{\tau-t} \int_{\mathcal{D}_\tau(i^t)} \mathsf{N}_\tau(i^\tau)^\gamma u\left(\frac{\mathsf{X}_\tau(i^\tau)}{\mathsf{N}_\tau(i^\tau)}\right) di^\tau \\ &= \sum_{\tau=t}^{\infty} \beta^{\tau-t} \int_{\mathcal{D}_\tau(i^t)} U^{Al}(\mathsf{N}_\tau(i^\tau), \mathsf{X}(i^\tau)) di^\tau,\end{aligned}$$

among all sequences $\left\{(\mathsf{X}_\tau, \mathsf{K}^o_{\tau+1}) : \mathcal{D}_\tau(i^t) \longrightarrow \mathbb{R}^4_+\right\}_{\tau\geq t}$ satisfying

$$\mathsf{C}^m_t(i^t) + b\mathsf{N}_{t+1}(i^t) + \mathsf{K}_{t+1}(i^t) = e_t;$$

for $\tau = t$; and

$$\int_{\mathcal{D}_\tau(i^t)} \left(\mathsf{C}^o_t(i^{\tau-1}) + \mathsf{C}^m_\tau(i^\tau) + b\mathsf{N}_\tau(i^\tau) + \mathsf{K}_{\tau+1}(i^\tau)\right) di^\tau \leq \int_{\mathcal{D}_\tau(i^t)} F_\tau(\mathsf{K}_\tau(i^{\tau-1}), \mathsf{N}_\tau(i^{\tau-1}))di^\tau.$$

for $\tau > t$. Therefore, when U^{Al} is concave, the optimization problem in the definition of $\mathcal{V}_t^D(e_t)$ is a standard, concave program with a concave objective function and a sequence of constraints that define a convex set. Thus, if $\mathcal{V}_t^D(e_t)$ is well defined for every $e_t \geq 0$, $\mathcal{V}_t^D$ must be concave, which taking into account that U^{Al} is concave, implies, in turn, that the allocation maximizing the utility of the dynasty head must be symmetric. If, in addition, U^{Al} is strictly concave, the solution to the optimization problem in the definition $\mathcal{V}_t(\overline{e}_0)$ must be unique, and the set of $\mathcal{A}-$efficient allocations reduces to a singleton.

Notice that the economy studied by Barro and Becker satisfies these conditions.

S.B.2 Concavity of value functions in an extension of Razin and Ben Zion´s (1976) model

But Alvarez's arguments cannot be applied to settings that extend the model of Razin and Ben Zion, in which

$$U^D(x_t, u^D_{t+1}) = u(x_t) + \beta u^D_{t+1},$$

where u is homothethic.

To be more precise, assume that production functions are time invariant –i.e., $F_t \equiv F$ $\forall t \geq 0$– and costs of rearing children are also time invariant and linear –i.e. $b_t(n_{t+1}) = bn_{t+1}$ $\forall t \geq 0$. In this setting, it is easy to see that each function W_t^D (and, hence, each function π_t) defined in the Supplementary Appendix S.A is both time invariant and separable. Specifically, for each $t \geq 0$ and every $(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}) \in \mathbb{R}^2_+ \times \mathbb{R}$, the function W_t^D can be written as

$$W_t^D(\widehat{e}_t, \widehat{e}_{t+1}, U^D_{t+1}) = W(\widehat{e}_t, \widehat{e}_{t+1}) + \beta U^D_{t+1},$$

where $W{:}\mathbb{R}^2_+ \longrightarrow \mathbb{R}$ is defined, for each $(\widehat{e}_t, \widehat{e}_{t+1}) \in \mathbb{R}^2_+$, by

$$W(\widehat{e}_t, \widehat{e}_{t+1}) = \max_{(x_t, k^o_{t+1})\in\Re^4_+} \left\{u(x_t) : c^m_t + b_t(n_{t+1}) + k^o_{t+1} \leq \widehat{e}_t; F(k^o_{t+1}, n_{t+1}) - c^o_{t+1} \geq n_{t+1}\widehat{e}_{t+1}\right\}.$$

More specifically, assume that the utility function u adopts the form

$$u(x_t) = \frac{1}{\theta}\,(v(x_t))^{\theta},$$

where $v : \mathbb{R} \to \mathbb{R}$ is non-decreasing, positive valued, concave, differentiable, and linearly homogeneous. Also, write $f(k^m)$ for $f(k^m) = F(k^m, 1)$ and assume that the following holds:

$$\lim_{k^m \longrightarrow 0} \left[(b + k^m)\, f'(k^m) - f(k^m)\right] > 0. \tag{S.12}$$

With these assumptions, capital *per worker* $k^o(e_t, e_{t+1})/n(e_t, e_{t+1})$ corresponding to a solution $x(e_t, e_{t+1})$ to the optimization problem in the definition of the indirect utility function $W(e_t, e_{t+1})$ depends only on e_{t+1}, that is

$$\frac{k^o(e_t, e_{t+1})}{n(e_t, e_{t+1})} = k^m(e_{t+1}).$$

Also, the indirect utility function adopts the form

$$W(e_t, e_{t+1}) = \frac{1}{\theta}\left(\frac{e_t}{\mathcal{E}(e_{t+1})}\right)^{\theta},$$

where $\mathcal{E} : \mathbb{R}_+ \to \mathbb{R}_+$ is the expenditure function defined, for each $e_{t+1} \in \mathbb{R}_+$, by

$$\mathcal{E}(e_{t+1}) = \min_{x_t \in \mathbb{R}^3_+} \left\{ c^m_t + \left(b + \frac{e_{t+1} - w(e_{t+1})}{R(e_{t+1})}\right) n_t + \left(\frac{1}{R(e_{t+1})}\right) c_t \;:\; v(x_t) \geq 1 \right\};$$

and the functions $R(\cdot)$ and $w(\cdot)$ are implicitly defined by

$$\mathcal{E}(e_{t+1}) = e_{t+1};$$

$$R(e_{t+1}) = f'_{t+1}(k^m(e_{t+1}));$$

and

$$w(e_{t+1}) = f(k^m(e_{t+1})) - k^m(e_{t+1}) f(k^m(e_{t+1})).$$

S.B.2.1 Necessary and sufficient conditions for concavity of the value function In the setting described, the value function $\mathcal{V}^D$ is also time invariant and satisfies, for each $e \in \mathbb{R}_+$, by

$$\mathcal{V}^D(e) = \max\left\{W(e, e') + \mathcal{V}^{Davg}(e') : e' \in \mathbb{R}_+\right\}.$$

where, in turn, $\mathcal{V}^{Davg}(e')$ –defined in (12) in the main paper– is the maximum average utility that the dynasty head obtains by providing her immediate descendants with an average income e', that is,

$$\mathcal{V}^{Davg}(e') = \max_{E:\mathbb{R}_+ \to [0,1] \in \triangle\mathbb{R}_+} \left\{ \int \mathcal{V}^D(e) dE(e) : \int dE(e) = e' \right\}.$$

Taking this into account, it is straightforward to show that $\mathcal{V}^D$ is concave if W is concave.

In turn, the concavity of the indirect utility function depends on the properties of the expenditure function $\mathcal{E}$ and, to be more precise, on the properties of the function $m : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined, for all $e \in \mathbb{R}_+$, by

$$m(e) = \frac{R(e)}{n(e,e)} = \frac{\mathcal{E}(e)}{e\mathcal{E}'(e)}.$$

Thus, m determines the inverse elasticity of current expenditures with respect to future incomes. It is straightforward to show that for any given growth path $\{e_t\}_{t\geq 0}$ we have

$$-\frac{D_1W(e_t,e_{t+1})}{D_2W(e_t,e_{t+1})} = \frac{R(e_{t+1})}{n(e_t,e_{t+1})\left(\frac{e_{t+1}}{e_t}\right)} = \left(\frac{e_{t+1}}{e_t}\right) m(e_{t+1}).$$

That is, in an efficient allocation, the ratio of the implicit interest rate to the rate at which total income grows is completely determined by $m(e_{t+1})$. Observe that a non-increasing m is a necessary condition for quasi-concavity of the function W. Also, it is straightforward to show that a sufficient condition ensuring that W is concave is that the function $l : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, defined, for each $e \in \mathbb{R}_+$, by $l(e) = \frac{\mathcal{E}'(e)}{\mathcal{E}'(e)} = em(e)$ is non-increasing.

S.B.2.2 Examples of non-concave indirect utility functions Unfortunately, the function m might be non monotonic. As we show in our previous work (see examples 2 and 3 in Conde-Ruiz et al, 2010) examples of a non-monotonic m can be easily obtained for utility and production functions of the form

$$v(x_t) = \begin{cases} \left[\gamma^m \left(c_t^m\right)^{\frac{\sigma-1}{\sigma}} + \gamma^o \left(c_{t+1}^o\right)^{\frac{\sigma-1}{\sigma}} + \gamma^n (n_{t+1})^{\frac{\sigma-1}{\sigma}}\right]^{\frac{\sigma}{\sigma-1}}, & \text{if } \sigma \neq 1, \\ \left(c_t^m\right)^{\gamma^m} \left(c_{t+1}^o\right)^{\gamma^o} (n_{t+1})^{\gamma^n}, & \text{if } \sigma = 1, \end{cases}$$

where and $\gamma^m + \gamma^o + \gamma^n = 1$; and

$$F(K,L) = \begin{cases} \left[A\,(K)^{\frac{\rho-1}{\rho}} + B\,(L)^{\frac{\rho-1}{\rho}}\right]^{\frac{\rho}{\rho-1}}, & \text{if } \rho \neq 1, \\ (K)^{\frac{1}{2}}\,(L)^{\frac{1}{2}}, & \text{if } \rho = 1. \end{cases}$$

Linear technologies. The simplest family of examples corresponds to economies in which the productions function adopts the linear form $F(k_t^o, n_t) = Rk_t^o + wn_t$, where R and w satisfy $bR > w$. This condition ensures that condition (S.12) is satisfied, which, in turn, implies that $\mathcal{E}(0) > 0$ is satisfied. In this case, the expenditure function $\mathcal{E}(\cdot)$ is concave and the indirect utility function W is quasiconvex. Moreover, the function m decreases if consumption goods are complements, that is, if $\sigma < 0$. For $\sigma > 0$, the function m is *single peak*, that is, there exists a unique point e^* such that m is decreasingly monotonic on $(0, e^*)$ and increasingly monotonic on (e^*, ∞).

Cobb Douglas preferences and non-linear production functions. When preferences adopt a

Cobb-Douglas form corresponding to $\sigma = 1$ we have

$$
\begin{aligned}
m(e) &= \left(\frac{\gamma^n e}{f'(k^m(e))[b+k^m(e)](\frac{\gamma^n}{\gamma^o+\gamma^n})}\right)^{-1} = \left(\frac{\gamma^n f(k^m(e)) - \gamma^n \frac{\gamma^o[f'(k^m(e)(b+k^m(e))]}{\gamma^o+\gamma^n}}{f'(k^m(e))[b+k^m(e)](\frac{\gamma^n}{\gamma^o+\gamma^n})}\right)^{-1} = \\
&= \left(\frac{(\gamma^o+\gamma^n) f(k^m(e))}{f'(k^m(e))[b+k^m(e)]} - \gamma^n\right)^{-1}
\end{aligned}
$$

In this case, the function $m(\cdot)$ might be increasing or decreasing depending on the properties of the production function. More specifically, for a CES production function we have

$$
\frac{f(k^m(e))}{f'(k^m(e))[b+k^m(e)]} = \frac{A\,(k^m(e))^{\frac{\rho-1}{\rho}} + B}{A\,(k^m(e))^{-\frac{1}{\rho}}\,[b+k^m(e)]} = \frac{k^m(e)}{[b+k^m(e)]} + \frac{B}{A}\frac{(k^m(e))^{\frac{1}{\rho}}}{[b+k^m(e)]},
$$

which, taking into account that $k^m(\cdot)$ is increasing, implies that $m(\cdot)$ is single peak (as defined above) if inputs are substitutes, i.e. $\rho > 1$.

S.B.2.3 Examples of non-concave value functions The non monotonicity of the function m might also induce non concavities in the unconstrained value function $\mathcal{V}^D$, that, in turn, may introduce differences between the unconstrained value function $\mathcal{V}^D$ and the constrained value function V^D. To see why, suppose m is single peak, that is, suppose there exists e^* such that m is decreasingly monotonic on the interval $[0, e^*)$ and increasingly monotonic on $[0, \infty)$. If $m(e^*) \leq 1$, there are two potential steady states of the dynastic maximization problem, e_s^1 and e_s^2 satisfying first order conditions for dynastic maximization, which, applied to a steady state e_s, reduce to $m(e_s) = \frac{1}{\beta}$. If both e_s^1 and e_s^2 satisfy both second order conditions –given by $D_1W(e_s, e) + \beta D_2W(e, e_s) < 0$ for each $e > e_s$– as well as transversality/dynamic efficiency conditions –given by $\pi(e_s) > 1$– for dynastic maximization. Both steady states e_s^1 and e_s^2 are, respectively, the solutions to the optimization problems in the definition of the –constrained– value functions $V^D(e_s^1)$ and $V^D(e_s^1)$. In this case, we can explicitly compute the constrained values $V^D(e_s^1)$ and $V^D(e_s^1)$ as

$$
V^D(e_s^1) = \frac{W(e_s^1, e_s^1)}{1-\beta} \text{ and } V^D(e_s^2) = \frac{W(e_s^2, e_s^2)}{1-\beta}.
$$

Suppose now that there exists $\lambda \in (0,1)$ satisfying, for a steady state $e_s \in \{e_s^1, e_s^2\}$

$$
W(e_s, \lambda e_s^1 + (1-\lambda)\, e_s^2) + \lambda\beta V^D(e_s^1) + (1-\lambda)\,\beta V^D(e_s^2) > \frac{W(e_s, e_s)}{1-\beta} = V^D(e_s). \qquad \text{(S.13)}
$$

Observe that the term at the left hand side of (S.13) is the welfare obtained by the dynasty head from her descendants if *a*) total income available to each immediate descendants is e_s; *b*) each immediate descendant provides with e_s^1 units of income to a proportion λ of her immediate descendants and with e_s^2 units of income to a proportion $(1-\lambda)$ of her immediate descendants; and, *c*) each of the two groups of descendants select the symmetric allocations that maximize the utility of the dynasty head among symmetric allocations. Thus, the term at the left hand side of (S.13) is the welfare obtained by the dynasty head with a particular feasible, non-symmetric allocation that provides the dynasty head with at most the same utility than the allocation that maximizes her utility among feasible allocations. Hence,

$$\mathcal{V}^D(e_s) \geq W(e_s, \lambda e_s^1 + (1-\lambda)\, e_s^2) + \lambda\beta V^D(e_s^1) + (1-\lambda)\,\beta V^D(e_s^2) > V^D(e_s), \qquad \text{(S.14)}$$

therefore establishing that $\mathcal{V}^D$ is not concave in a neighborhood of e_s.

Inequality (S.14) holds for many preferences and production functions for which the function m is single peak –in the sense described above. More precisely, for Cobb-Douglas preferences and a CES technology, if

$$\theta = -0.3; b = 1;\ \beta = 0.5;\ v(x_t) = \left[c_t^m c_{t+1}^o n_t\right]^{\frac{1}{3}};\ \text{and, } F(k_{t+1}^o, n_{t+1}) = \left([k_{t+1}^o]^{\frac{1}{2}} + 3[n_{t+1}]^{\frac{1}{2}}\right)^2,$$

then $e_s^1 = 7.07$, $e_s^2 = 127.33$ and (S.14) holds for $e_s = e_s^2$ and $\lambda > 0.66$. In the case that both preferences and technology are represented by CES function, if $b = 1$; $\beta = 0.71$;

$$\begin{aligned} \theta = -1;\ b = 1;\ \beta = 0.4;\ v(x_t) &= \frac{1}{3^2}\left[(c_t^m)^{\frac{1}{2}} + \left(c_{t+1}^o\right)^{\frac{1}{2}} + (n_t)^{\frac{1}{2}}\right]^2;\ \text{ and,} \\ F(k_{t+1}^o, n_{t+1}) &= \left([k_{t+1}^o]^{\frac{1}{2}} + 3[n_{t+1}]^{\frac{1}{2}}\right)^2; \end{aligned}$$

then $e_s^1 = 4.30$, $e_s^2 = 30.63$ and (S.14) holds for $e_s = e_s^2$ and $\lambda > 0.08$.

The intuition of why non-convexities in the feasible set might generate non-concavities of value functions is the following: when W is not quasiconcave, the function defined by $H(e) = W(e, e)$ is not increasingly monotonic. More precisely, in the examples given above, H reaches a local maximum at the point $e^* > e_s^1$ satisfying $m(e^*) = 1$ and a local minimum at the point $e_* \in (e^*, e_s^2)$, for which $m(e_*) = 1$ is also satisfied. Thus, even though transversality conditions impose that $W(e_s^2, e_s^2) > W(e_s^1, e_s^1)$ is satisfied, the difference $W(e_s^2, e_s^2) - W(e_s^1, e_s^1)$ is small.

Further, it can be shown that, when this occurs, the highest steady state e_s^2 is unstable. That is, the policy function, that is, the function $P : \mathbb{R}_+ \to \mathbb{R}_+$ satisfying, for each $e \in \mathbb{R}_+$, $V^D(e) = W(e, P(e)) + V^D(P(e))$ crosses the line defined by $e_{t+1} = e_t$ from below. Hence, if the dynasty head initial wealth is close to e_s^2, she is almost indifferent between 1) staying at the steady state e_s^2; 2) initiating a sustained growth path for which the income available to each of her descendants is higher and higher and fertility rates are lower and lower; or, 3) initiating a growth path for which the wealth available to each of her descendants is lower and lower and fertility rates are higher and higher, driving the economy towards the steady state e_s^1. Since consumption goods and the number of children are substitutes, the welfare obtained by the dynasty head with either one of these three alternatives is (almost) the same. But, if the dynasty head is not restricted to treat all of her descendants symmetrically, then she can do even better by providing some of their descendants with a relatively low income e_1^s and some of their descendants with a high income e_s^2.

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