ECOBAS Working Papers

2016 - 12

Title:

CONSISTENCY IN PERT PROBLEMS

Authors:

Gustavo Bergantiños
Universidade de Vigo

Alfredo Valencia-Toledo
Universidade de Vigo

Juan Vidal-Puga
Universidade de Vigo

Economics and Business Administration for Society

www.ecobas.es

Consistency in PERT problems

Gustavo Bergantiños and Alfredo Valencia-Toledo and Juan Vidal-Puga

Universidade de vigo

2016

Consistency in PERT problems*

G. Bergantiños†[1,3], A. Valencia-Toledo‡[2,3] and J. Vidal-Puga§[1,3]

[1]Department of Statistic and Operational Research
[2]Research Group in Economic Analysis
[3]Universidade de Vigo, Spain

January 21, 2016

Abstract

The Program Evaluation Review Technique (PERT) is a tool used to schedule and coordinate activities in a complex project. In assigning the cost of a potential delay, we characterize the Shapley rule as the only rule that satisfies consistency and other desirable properties.

Keywords— PERT problem, consistency, delay.

1 Introduction

Large-scale projects require the coordination of numerous activities, some of which can be performed sequentially, while others can take place in parallel. The Program Evaluation and Review Technique (PERT) is an operational research tool used to schedule and coordinate activities. The tool was developed by the U.S. Navy in the late 1950s to manage the Polaris submarine

*A. Valencia-Toledo thanks the Ministry of Education of Peru for its financial support through the grant "Beca Presidente de la República del Programa Nacional de Becas y Crédito Educativo (PRONABEC)". G. Bergantiños and J. Vidal-Puga acknowledge financial support from the Spanish Ministerio de Ciencia e Innovación through grant ECO2011-23460.

†E-mail: gbergant@uvigo.es
‡E-mail: alfredo.valencia@uvigo.es
§E-mail: vidalpuga@uvigo.es

missile program, a project involving thousands of contractors.[1] The tool uses networks to model all the activities and can help reduce both the time and cost required to finish a project.

PERT planning involves several steps, including estimating of the time required for each activity and determining the *critical activities* (activities that have the potential to delay the entire project). In the example shown in Figure 1, each arc represents an activity and the number in brackets the corresponding time. The critical activities are a, b, c, d and g.

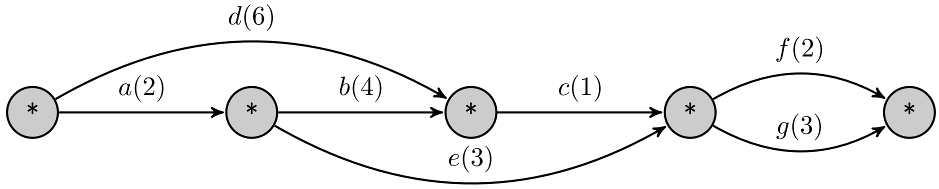

Figure 1: A PERT graph.

For projects that suffer from delays, it may be useful to identify the activities involved and the responsibility of each for causing the delay. Although costs typically arise when a project is not fully completed on time, it may also be the case these can be reduced by ensuring certain important parts of the project are completed.

In this paper, we seek to contribute to the literature on cost allocation by studying the share of the cost of the delay between activities. The subject was first studied by Bergantiños and Sánchez (2002) and Brânzei et al. (2002). The former present two rules, one based on serial cost sharing problems and the other on game theory. They also introduce some desirable properties and study which of these are satisfied by the rules. The second study takes a different approach, assigning two related problems to each PERT situation: a bankruptcy problem and a cooperative game. The rules and values suggested in the respective fields (e.g. the proportional rule and the Shapley value) are then studied and applied to the original PERT

[1] Another similar technique is the Critical Path Method (CPM) which was developed in 1957 for project management in the private sector. CPM and PERT have become synonyms with a number of variations used to refer to the same technique (PERT, CPM, or PERT/CPM).

situation.

Furthermore, Bergantiños and Vidal-Puga (2009) use an indirect approach to determine a rule that distributes the PERT time equally among the activities.

In the literature on cost allocation, one of the most important issues is to find allocation rules that are optimal, fair, and strategically stable. In PERT situations, however, strategic stability has no clear definition: the graph structure is closed and, as such, no group of activities can do without the others. This means that concepts such as the core and population monotonicity are applicable.

As such, we must focus on properties based on the ideas of optimality and fairness. Examples of properties based on optimality are efficiency (the cost assigned should exactly cover the delay cost) and cost monotonicity (no activity should pay more if it finishes ahead of schedule). Examples of properties based on fairness are dummy (an activity finishing on schedule pays nothing), anonymity (the allocation does not depend on the name of the activities), and symmetry (symmetric activities pay the same). In general there are many rules to satisfy these properties.

As Winter (2002) argues, one of the fundamental requirements of any legal system is that it is internally consistent. In terms of the rules we would like to use for allocating the costs of delays, the allocation will typically depend on the PERT graph, scheduled completion times, actual completion times and (as argued above) the cost function.

Let us assume we focus our attention on a group of activities B. Once the project is finished, we can attempt to evaluate the cost of these activities if they had been completed at different times. This evaluation gives rise to a new cost function for B and hence a different (reduced) PERT situation. The rule is said to be *consistent* if the payoff for the activities in the reduced PERT situation are the as in the original PERT situation. In this paper, we define a rule for PERT problems and show that it is consistent in the sense explained above.

The paper is organized as follows: in Section 2, we give the notation and define the model; in Section 3 we study the Shapley rule and provide a characterization based on anonymity, consistency, scale invariance, and standard for two; finally, in Section 4, we study other rules.

2 The problem

We denote the set of nonnegative real numbers as $\mathbb{R}_+$ and the set of natural numbers as $\mathbb{N}$. Given a finite set N, we denote the cardinal set of N as $|N|$. Given $x, y \in \mathbb{R}^N$, we write $x \leq y$ when $x_i \leq y_i$ for all $i \in N$. Given $x \in \mathbb{R}^N$ and $S \subset N$, we denote the restriction of x on S as x_S, *i.e.* $x_S = (x_i)_{i \in S}$. Given $x \in \mathbb{R}^N$, $S \subset N$ and $y \in \mathbb{R}^S$, x/y denotes the vector $z \in \mathbb{R}^N$ with $z_i = y_i$ if $i \in S$ and $z_i = x_i$ if $i \in \overline{S}$ where $\overline{S}$ denotes the set $N \setminus S$. For simplicity when $S = \{i\}$ and $y = (y_i) \in \mathbb{R}^S$, we write x/y_i instead of x/y. 0_N denotes the vector $(0, ..., 0) \in \mathbb{R}^N_+$.

Let Π_N be the set of all permutations over the finite set $N \subset \mathbb{N}$. Given $\pi \in \Pi_N$, let $Pre\,(i, \pi)$ denote the set of elements of N which come before i in the order given by π, *i. e.* $Pre\,(i, \pi) = \{j \in N \mid \pi\,(j) < \pi\,(i)\}$. Given $S \subset N$, let π_S denote the order induced by π among agents in S.

A game with transferable utility (*TU* game) is a pair (N, v) where N is a finite set and $v : 2^N \to \mathbb{R}$ is the *characteristic function* satisfying $v(\emptyset) = 0$. Given $S \subset N$, we denote the *restricted game* of (N, v) to S as (S, v'), where $v'(T) = v(T)$ for all $T \subset S$. We denote by $Sh\,(N, v)$ the *Shapley value* (Shapley, 1953) of the *TU* game (N, v). It is well known that for all $i \in N$,

$$Sh_i\,(N, v) = \frac{1}{|N|!} \sum_{\pi \in \Pi_N} \left(v\,(Pre\,(i, \pi) \cup \{i\}) - v\,(Pre\,(i, \pi))\right).$$

A *PERT problem with delays* over N is a tuple $P = \left(N, \prec, x^0, x, C\right)$ where

- $N \subset \mathbb{N}$ denotes a finite set of activities forming the project.
- $\prec$ is a *partial order* in N, *i.e.* $\prec$ satisfies *transitivity* ($i \prec j$ and $j \prec k$ implies $i \prec k$) and *asymmetry* ($i \prec j$ implies $j \nprec i$). Given $i, j \in N$, $i \prec j$ means that activity j cannot begin until activity i is finished.
- $x^0 \in \mathbb{R}^N_+$ and for each $i \in N$, x^0_i is the *expected completion time* of activity i.

- $x \in \mathbb{R}_+^N$ and for each $i \in N$, x_i is the *actual completion time* of activity i. We assume that $x \geq x^0$.

- $C : \mathbb{R}_+^N \to \mathbb{R}_+$ is the *cost function*. We assume that C is a nondecreasing function in each coordinate satisfying that $C\left(x^0\right) = 0$. Given $x \in \mathbb{R}_+^N$, $C\left(x\right)$ denotes the cost associated with the delay in the project.

A *path* $\gamma = \{i_1, \ldots, i_p\} \subset N$ in P is a collection of activities such that $i_k \prec i_{k+1}$ for each $k \in \{1, \ldots, p-1\}$. Let Γ^P be the set of all paths in P. An *expected critical path* in P is a path $\gamma^0 \in \Gamma^P$ such that $\sum_{i\in\gamma^0} x_i^0 = \max_{\gamma\in\Gamma^P} \sum_{i\in\gamma} x_i^0$. An *actual critical path* in P is a path $\gamma^1 \in \Gamma^P$ such that $\sum_{i\in\gamma^1} x_i = \max_{\gamma\in\Gamma^P} \sum_{i\in\gamma} x_i$.

We define the *PERT delay* of the project P as $d(P) = \max_{\gamma\in\Gamma^P} \sum_{i\in\gamma} x_i - \max_{\gamma\in\Gamma^P} \sum_{i\in\gamma} x_i^0$. We define the *delay* of activity $i \in S$, d_i, as the difference between its actual completion time and its expected completion time, *i.e*, $d_i = x_i - x_i^0 \geq 0$.

Typically, the cost function only depends on the PERT delay of the project i.e. there exists $f : \mathbb{R}_+ \to \mathbb{R}_+$ with $f(0) = 0$ and such that $C(y) = f(d(N, \prec, x^0, y, C))$ for all $y \geq x^0$.

Example 1 *Assume that we have to carry out a project with three activities. The PERT situation is given by* $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$ *where* $N = \{1, 2, 3\}$, $1 \prec 2$ *and* $1 \prec 3$, $x^0 = (10, 20, 30)$, $x = (20, 40, 40)$, *and* $C(x_1, x_2, x_3) = x_1 + \max\{x_2, x_3\} - 40$. *This situation is depicted in Figure 2.*

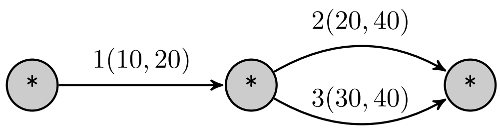

Figure 2: Graph of PERT problem with delays.

We note that there is a unique expected critical path $\gamma^0 = \{1, 3\}$ *and there are two actual critical paths* $\gamma_1^1 = \{1, 2\}$ *and* $\gamma_2^1 = \{1, 3\}$. *We obtain the PERT delay of the project as* $d(P) = 60 - 40 = 20$. *Note that there exists a function* $f : \mathbb{R}_+ \to \mathbb{R}_+$ *such that* $f(t) = t$ *for all* $t \in \mathbb{R}_+$ *which generates* *the cost* $C(x) = 20 + \max\{40, 40\} - 40 = 20$ *and the delay of each activity is given by* $d = (10, 20, 10)$*. Also, note that activity* 2 *can delay up to* 10 *without generating delay cost in the project. Moreover, there exists a positive delay cost of the project when either any activity in* γ^0 *has a positive delay, activity* 2 *has a delay greater than* 20 *or all activities have positive delays.*

Given $S \subset N$, we denote $\prec_S$ the restriction of $\prec$ on S, *i.e.* $i \prec_S j$ iff $i \prec j$, for all $i, j \in S$.

We denote the set of PERT problems with delay over a finite set N as $\mathcal{P}(N)$. We also denote the set of all PERT problems with delay as $\mathcal{P}$, *i.e.* $\mathcal{P} = \bigcup_{N \subset \mathbb{N}, |N| < \infty} \mathcal{P}(N)$.

A *rule* is a function ϕ on $\mathcal{P}$ that assigns to each $P \in \mathcal{P}(N)$ a vector $\phi(P) \in \mathbb{R}^N_+$ satisfying:

Efficiency (EF) $\sum\limits_{i \in N} \phi_i(P) = C(x)$ and

Null Delay (ND) $\phi_i(P) = 0$ when $x_i = x_i^0$.

$C(x)$ is the total cost produced by the delay of the project. Given an activity $i \in N$, $\phi_i(P)$ represents the amount paid by activity i. Efficiency says that the total cost must be distributed among the activities involved in the project. Null delay says that activity i must pay nothing when it has no delay.

In this paper we restrict to rules satisfying the following properties:

Additivity (AD) if for all pair of problems $P = \left(N, \prec, x^0, x, C\right)$ and $P' = \left(N, \prec, x^0, x, C'\right)$,

$$\phi\left(N, \prec, x^0, x, C + C'\right) = \phi(P) + \phi\left(P'\right)$$

where $(C + C')(y) = C(y) + C'(y)$ for all $y \geq x^0$.

Dummy (DU) if for all $P = \left(N, \prec, x^0, x, C\right)$ and $i \in N$ such that $C(y) = C\left(y / x_i^0\right)$ for all $y \geq x^0$,

$$\phi_i(P) = 0.$$

Additivity is a properties that comes naturally in PERT problem situations. In order to illustrate this property, we consider a house build project. Since there exits a total delay cost of the project, we can distinguish between two cost functions that are generated by two different reasons. One of them is a variable cost C (for example, the rent payment in another house) and the other one is a fixed cost C' (for example, the payment of a contractual penalty). Assume that activities in N agree that ϕ is the right solution. They can proceed in two ways. First, they can apply ϕ in each of the two PERT problems with delays given by C and C'. Second, they can apply ϕ only once in the PERT problem with delays with the cost function given by $C + C'$. If both procedures give the same result, we say that ϕ is additive.

Dummy is another property that comes naturally in PERT problem situations. When there exists an activity that have a delay independent of the total delay of the project, this activity pays nothing.

We now define two extreme subclasses of PERT problems with delays. In the first all the activities are related whereas in the second activities are unrelated.

We say that P is *sequential* if activities are lined up, *i.e.* $N = \{i_k\}_{k=1}^{n}$ where $i_k \prec i_{k+1}$ for all $k \in \{1, ..., n-1\}$.

We say that P is *parallel* if activities can be done simultaneously, *i.e.* for all $i, j \in N$, $i \nprec j$.

The previous PERT problems with delays generalize two well-known problems of the literature of cost sharing: the serial cost sharing problem and the airport problem.

Following Moulin and Shenker (1992) a *serial cost sharing problem* is a triple (N, q, c) where N is the set of agents, $q = (q_i)_{i \in N} \in \mathbb{R}_+^N$ and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function satisfying $c(0) = 0$.

This model has several interpretations. We present the interpretation based on cost sharing. Each agent $i \in N$ demands an amount $q_i \in \mathbb{R}_+$ of a perfectly divisible good. The total amount required by the agents, $\sum_{i \in N} q_i$, is produced. The cost $c\left(\sum_{i \in N} q_i\right)$ is divided among the agents.

Consider a PERT problem with delays where activities are lined up and the cost of delaying the project depends only on the total delay of the project (the difference between the actual completion time and the expected completion time). Formally, $P = \left(N, \prec, x^0, x, C\right)$ where P is sequential and there exists a nondecreasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $C(y) = c\left(\sum_{i \in N}(y_i - x_i^0)\right)$ for all $y \geq x^0$.

This PERT problem with delays can be considered as a serial cost sharing problem where the set of agents are the activities and the demands are the delays.

Reciprocally, given a serial cost sharing problem (N, q, c), we can associate a sequential PERT situation with delays $\left(N, \prec, x^0, x, C\right)$ where the set of activities is N either $i \prec j$ or $j \prec i$ for all $i, j \in N$, and $y = x^0 + q$ for some $x^0 \in \mathbb{R}^N$, and $C(y) = c\left(\sum_{i \in N}(y_i - x_i^0)\right)$ for all $y \geq x^0$.

Thus, serial cost sharing problems can be considered as a subclass of sequential PERT situations with delay. When we identify a serial cost sharing problem with a PERT problem with delays we are thinking, in both cases, in how to divide the cost in a fair way among the agents or activities.

Following Littlechild and Owen (1973), an *airport problem* is a pair (N, q, c) where N is the set of agents, $q \in \mathbb{R}_+^N$, and $c : \mathbb{R}_+ \to \mathbb{R}_+$ is a nondecreasing function satisfying that $c(0) = 0$. Even though the mathematical model is the same as in serial cost sharing problems, the situation is different. N represents several kinds of aircraft landings in some airport. Each aircraft needs, for landing, a runway of length at least q_i. We have to construct a runway of length at least $\max_{i \in N}\{q_i\}$. When an aircraft lands in this airport, it must pay a tariff, which depends, among other things, of the construction cost of the runway. The problem is to determine this part of the tariff in a fair way.

Consider a PERT problem with delays where activities are parallel and the cost of delaying the project depends only on the total delay of the project. Formally, $P = \left(N, \prec, x^0, x, C\right)$ where P is parallel and there exists a nondecreasing function $c : \mathbb{R}_+ \to \mathbb{R}_+$ such that $C(y) = c\left(\max_{i \in N}\left\{y_i - x_i^0\right\}\right)$ for all $y \geq x^0$.

This PERT problem with delays can be considered as an airport problem where the agents are the activities and the demand of each agent is its delay.

Reciprocally, given an airport problem (N, q, c) we can associate a parallel PERT problem with delays $\left(N, \prec, x^0, x, C\right)$ where P is parallel, $x - x^0 = q$ for some $x^0 \in \mathbb{R}^N$, and $C(y) = c\left(\max_{i \in N}\left\{y_i - x_i^0\right\}\right)$ for all $y \geq x^0$.

Thus, airport problems can be considered as a subclass of PERT problems with delays.

A *cost sharing problem* is a triple (N, q, c) where N is the finite set of agents, $q = (q_i)_{i \in N} \in \mathbb{R}_+^N$ and $c : \mathbb{R}_+^N \to \mathbb{R}_+$ is a nondecreasing function satisfying that $c(0_N) = 0$. Notice that serial cost sharing problems and airport problems are particular classes of cost sharing problems.

Moreover, cost sharing problems can be considered, from a mathematical point of view, as a particular subclass of PERT problems with delays. Just assume that $\prec$ is empty, *i.e.* activities are unrelated.

3 The Shapley rule

The Shapley value (Shapley, 1953) is defined for general cooperative games, but it has been successfully applied to many particular situations, such as voting games (Shapley and Shubik, 1954), cost sharing problems (Shubik, 1962), serial cost problems (Moulin and Shenker, 1992) and airport landing problems (Littlechild and Owen, 1973), just to mention a few.

We now introduce a rule for PERT problems with delays based on the Shapley value.

Given a PERT problem with delays $P = \left(N, \prec, x^0, x, C\right)$ we associate a *TU* game $\left(N, v^P\right)$ where $v^P(S) = C\left(x^0/x_S\right)$ for all $S \subset N$. Notice that $v^P(S)$, represents the cost of delaying the project when activities of $\overline{S}$ have not delay and activities of S have a delay given by $x - x^0$.

Given $P \in \mathcal{P}$ we define the *Shapley rule* φ as

$$\varphi(P) = Sh\left(N, v^P\right).$$

It is straightforward to prove that φ is a well-defined rule, i.e. it satisfies EF and ND. Moreover, it also satisfies ADD and DU.

We now introduce some other properties of rules. The first five properties are quite standard in the literature. The last property is specific of PERT problems with delays.

Given $\pi \in \Pi_N$ and $y \in \mathbb{R}_+^N$ we define $\pi(y) \in \mathbb{R}_+^N$ as the vector where $\pi(y)_i = y_i$ for all $i \in N$. Given $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$ and $\pi \in \Pi_N$, we define the PERT problem with delays $P^\pi = \left(N, \prec_\pi, \pi\left(x^0\right), \pi(x), C^\pi\right)$

where $i \prec_\pi j$ iff $\pi(i) \prec \pi(j)$ for all $i, j \in N$, and $C^\pi(y) = C\left(\pi^{-1}(y)\right)$ for all $y \in \mathbb{R}_+^N$.

Anonymity (AN) ϕ satisfies AN if for all $P \in \mathcal{P}$, $\pi \in \Pi_N$, and $i \in N$,

$$\phi_i(P) = \phi_{\pi(i)}(P^\pi).$$

Anonymity says that cost shares should not depend on the name of the activities.

Given $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$, $S \subset N$ and $y \in \mathbb{R}^S$ we define the *reduced problem* associated with S and the rule ϕ as $P^{\phi,S} = \left(S, \prec_S, x_S^0, x_S, C^{\phi,S}\right)$ where $C^{\phi,S} : \mathbb{R}_+^S \rightarrow \mathbb{R}_+$ is defined as

$$C^{\phi,S}(y) = C(x/y) - \sum_{i \in \overline{S}} \phi_i\left(N, \prec, x^0, x/y, C\right).$$

Consistency $(CONS)$ ϕ satisfies $CONS$ if for all $S \subset N$ we have

$$\phi_S(P) = \phi\left(P^{\phi,S}\right).$$

Given $S \subset N$, assume that activities in $\overline{S}$ agree that ϕ is the right solution. Activities in S can proceed in two ways. First, they can divide the cost according with ϕ. Second, they can construct a reduced problem among themselves assuming that activities in $\overline{S}$ will pay according with ϕ. If both procedures give the same result for activities in S, we say that ϕ is consistent.

Consistency is a well-known principle in the literature. See, for instance, the survey of Thomson (2009).

Monotonicity (MON) ϕ satisfies MON if for all $P = \left(N, \prec, x^0, x, C\right)$, $i \in N$, and $P' = \left(N, \prec, x^0, x/x'_i, C\right)$ with $x_i \leq x'_i$,

$$\phi_i(P) \leq \phi_i\left(P'\right).$$

If the delay of activity i increases, the amount paid by this activity cannot decrease.

Given $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$ and $\lambda \in \mathbb{R}^N_{++}$, we define the cost function $C^\lambda : \mathbb{R}^N_+ \to \mathbb{R}_+$ such that $C^\lambda(\lambda y) = C(y)$ for all $y \in \mathbb{R}^N_+$, where $\lambda y = (\lambda_i y_i)_{i \in N}$.

Scale Invariance (SI) ϕ satisfies SI if for all $P = \left(N, \prec, x^0, x, C\right)$ and $\lambda \in \mathbb{R}^N_{++}$ we have that

$$\phi\left(N, \prec, \lambda x^0, \lambda x, C^\lambda\right) = \phi\left(N, \prec, x^0, x, C\right).$$

Changing the units in which the delays are measured will not affect the cost shares.

Standard for Two (ST) ϕ satisfies ST if given $P = \left(N, \prec, x^0, x, C\right)$ where N has two activities, say i and j,

$$\phi_i(P) = C\left(\left(x_i, x_j^0\right)\right) + \frac{1}{2}\left(C\left((x_i, x_j)\right) - C\left(\left(x_i, x_j^0\right)\right) - C\left(\left(x_i^0, x_j\right)\right)\right).$$

Each activity pays the cost caused by its own delay (assuming that the other has not delay). Moreover, the difference between the total cost of delaying the project and the sum of the cost caused by both activities, is divided equally among them.

ST is inspired by the property of standard for two of TU games defined in Hart and Mas-Colell (1989).

We say that $x_i \in \mathbb{R}_+$ constitutes an *irrelevant completion time for activity* $i \in N$ for $(N, \prec, C)$ if for all $x' \in \mathbb{R}^N_+$ with $x' \geq x^0$,

$$C\left(x'/x_i\right) = C\left(x'/x_i^0\right).$$

This means that, independently of the delays of the other activities, if activity i has a delay $x_i - x_i^0$, the total cost of the project is the same as if activity i has not delay.

Independence of Irrelevant Delays (IID) ϕ satisfies IID if given $P = \left(N, \prec, x^0, x, C\right)$, $i \in N$, and x_i is an irrelevant completion time for activity i,

$$\phi_i(P) = 0.$$

If the delay of activity i does not affect the total cost of the project, activity i pays nothing.

Example 2 *We have to carry out a project of two activities (*1 *and* 2*), which can be done independently. The expected completion time of activities* 1 *and* 2 *are* 10 *and* 5 *respectively. Thus, the total expected time for finishing the project is* 10*. Moreover, we have to pay* 100 *for each unit of time that the project is delayed. In this example,* $N = \{1, 2\}$*,* $\prec$ *is empty,* $x^0 = (10, 5)$*, and* $C(x) = 100 (\max\{x_1, x_2\} - 10)$*. It is not difficult to check that* x_2 *is an irrelevant completion time for activity* 2 *when* $x_2 \leq 10$*. Assume that activities* 1 *and* 2 *have a delay of* 2 *and* 4*, respectively. The cost of delaying the project is* 200*. Thus, if a rule satisfies* IID*, all the cost should be paid by activity* 1*.*

Some well known rules do not satisfy IID, as we will see later.

Even though IID has the same flavor as ND, they are different properties. IID implies ND but the reciprocal is not true.

We now prove that the Shapley rule satisfies all the properties mentioned before.

Theorem 1 *The Shapley rule satisfies* AN*,* $CONS$*,* MON*,* SI*,* ST*, and* IID*.*

Proof. We first prove that φ satisfies AN. Take $P = (N, \prec, x^0, x, C) \in \mathcal{P}$, $\pi \in \Pi_N$, P^π, and $S \subset N$. Thus,

$$v^{P^\pi}(\pi(S)) = C^\pi\left(\pi(x)/\pi\left(x^0\right)_S\right) = C\left(x/x^0_S\right) = v^P(S).$$

Since the Shapley value satisfies anonymity in TU games, we conclude that φ satisfies AN.

We now prove that φ satisfies $CONS$. Given $P = (N, \prec, x^0, x, C) \in \mathcal{P}$ and $S \subset N$, we should prove that $\varphi_S(P) = \varphi\left(P^{\varphi,S}\right)$ or, equivalently,

$$Sh_S\left(N, v^P\right) = Sh\left(S, v^{P^{\varphi,S}}\right).$$

Hart and Mas-Colell (1989) proved that if (N, v) is a cooperative game and $S \subset N$, then $Sh_S(N, v) = Sh(S, v_S)$, where for all $T \subset S$,

$$v_S(T) = v\left(T \cup \overline{S}\right) - \sum_{i \in \overline{S}} Sh_i\left(T \cup \overline{S}, v\right).$$

Hence, it is enough to prove that $(v^P)_S = v^{P^{\varphi,S}}$. Since Sh satisfies efficiency in TU games, for all $T \subset S$,

$$\left(v^P\right)_S(T) = \sum_{i \in T} Sh_i\left(T \cup \overline{S}, v^P\right).$$

Since φ satisfies EF and ND,

$$\begin{aligned} v^{P^{\varphi,S}}(T) = C^{\varphi,S}\left(x_S^0/x_T\right) &= \sum_{i \in S} \varphi_i\left(N, \prec, x^0, x/\left(x_S^0/x_T\right), C\right) \\ &= \sum_{i \in S} \varphi_i\left(N, \prec, x^0, x/x_{S\setminus T}^0, C\right) \\ &= \sum_{i \in T} \varphi_i\left(N, \prec, x^0, x/x_{S\setminus T}^0, C\right). \end{aligned}$$

We denote $P' = \left(N, \prec, x^0, x/x_{S\setminus T}^0, C\right)$. Thus,

$$v^{P^{\varphi,S}}(T) = \sum_{i \in T} Sh_i\left(N, v^{P'}\right).$$

Now, it is enough to prove that $Sh_T\left(N, v^{P'}\right) = Sh_T\left(T \cup \overline{S}, v^P\right)$. We prove it in two steps.

1. We first prove that $Sh_T\left(N, v^{P'}\right) = Sh_T\left(T \cup \overline{S}, v^{P'}\right)$. By the carrier property of the Shapley value (Shapley, 1953), it is enough to prove that $T \cup \overline{S}$ is a carrier for $\left(N, v^{P'}\right)$, *i.e.* $v^{P'}(R) = v^{P'}\left(R \cap \left(T \cup \overline{S}\right)\right)$ for all $R \subset N$. Given $R \subset N$,

$$v^{P'}(R) = C\left(x^0/\left(x/x_{S\setminus T}^0\right)_R\right) = C\left(x^0/x_{R \cap \overline{(S\setminus T)}}\right)$$

   and

$$\begin{aligned} v^{P'}\left(R \cap \left(T \cup \overline{S}\right)\right) &= C\left(x^0/\left(x/x_{S\setminus T}^0\right)_{R \cap \left(T \cup \overline{S}\right)}\right) \\ &= C\left(x^0/x_{R \cap \left(T \cup \overline{S}\right)}\right). \end{aligned}$$

   Since $\overline{(S \setminus T)} = T \cup \overline{S}$, the result holds.

2. We now prove that $Sh_T\left(T \cup \overline{S}, v^{P'}\right) = Sh_T\left(T \cup \overline{S}, v^P\right)$. We prove that $v^{P'} = v^P$. Given $R \subset T \cup \overline{S}$,

$$v^{P'}(R) = C\left(x^0/\left(x/x_{S\setminus T}^0\right)_R\right) = C\left(x^0/x_R\right) = v^P(R).$$

We now prove that φ satisfies MON. Let $P = \left(N, \prec, x^0, x, C\right)$, $i \in N$, and $P' = \left(N, \prec, x^0, x/x'_i, C\right)$ with $x_i \leq x'_i$. Given $S \subset N$, $i \notin S$,

$$\begin{aligned} v^P\left(S \cup \{i\}\right) - v^P\left(S\right) &= C\left(x^0/x_{S\cup\{i\}}\right) - C\left(x^0/x_S\right) \\ &\leq C\left(x^0/\left(x_S, x'_i\right)\right) - C\left(x^0/x_S\right) \\ &= v^{P'}\left(S \cup \{i\}\right) - v^{P'}\left(S\right). \end{aligned}$$

Since $\varphi\left(P\right) = Sh\left(N, v^P\right)$ and $\varphi\left(P'\right) = Sh\left(N, v^{P'}\right)$, φ satisfies MON.

We now prove that φ satisfies SI. Take $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$, $\lambda \in \mathbb{R}^N_{++}$, $P^\lambda = \left(N, \prec, \lambda x^0, \lambda x, C^\lambda\right)$, and $S \subset N$. Thus,

$$v^{P^\lambda}\left(S\right) = C^\lambda\left(\left(\lambda_i x_i^0\right)_{i\in\overline{S}}, \left(\lambda_i x_i\right)_{i\in S}\right) = C\left(x^0/x_S\right) = v^P\left(S\right).$$

Now, it is trivial to see that φ satisfies SI.

Since $\varphi\left(P\right)$ is the Shapley value of v^P, it is trivial to see that φ satisfies ST.

We now prove that φ satisfies IID. Given $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$, let x_i be an irrelevant completion time for activity i. Assume that $S \subset N$ and $i \notin S$. Then,

$$v^P\left(S \cup \{i\}\right) = C\left(x^0/x_{S\cup\{i\}}\right) = C\left(x^0/x_S\right) = v^P\left(S\right).$$

Since $\varphi\left(P\right) = Sh\left(N, v^P\right)$, φ satisfies IID. ■

Theorem 2 *There is a unique rule satisfying AN, CONS, SI, and ST.*

Proof. Let ϕ be a rule satisfying these properties. Fix a PERT problem with delays $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$. We define the partial order $\prec^*$ in $\mathbb{N}$ such that $i \prec^* j$ if and only if $i, j \in N$ and $i \prec j$. Let $\mathcal{P}'$ be the class of PERT problems with delays satisfying that $P' = \left(N', \prec', x^{0\prime}, x', C'\right) \in \mathcal{P}'$ if and only if $\prec'_{N'} = \prec^*_{N'}$, and $x_i^{0\prime} = x_i^0$ for all $i \in N \cap N'$ and $x_i^{0\prime} = 0$ otherwise. Given $P' = \left(N', \prec', x^{0\prime}, x', C'\right) \in \mathcal{P}'$ we can associate a cost sharing problem $\left(N', q', c'\right)$ where $q'_i = x'_i - x_i^{0\prime}$ for all $i \in N'$ and $c'\left(q\right) = C'\left(x^{0\prime} + q\right)$ for all $q \in \mathbb{R}^{N'}_+$. Reciprocally, given a cost sharing problem $\left(N', q', c'\right)$, we can associate a PERT problem with delays $P' = \left(N', \prec', x^{0\prime}, x', C'\right) \in \mathcal{P}'$ where $x'_i = x_i^{0\prime} + q_i$ for all $i \in N \cap N'$ and $x'_i = q_i$ otherwise, and $C'\left(y\right) = c'\left(y - x^{0\prime}\right)$ for all $y \geq x^{0\prime}$. This correspondence is one-to-one because both $\prec'$ and $x^{0\prime}$

are fixed. Thus, we write $\left(N', \prec', x^{0\prime}, x', C'\right)$ for PERT problems with delays and (N', q', c') for cost sharing problems. We define a rule ψ in cost sharing problems through ϕ. For each cost sharing problem (N', q', c'),

$$\psi\left(N', q', c'\right) = \phi\left(N', \prec', x^{0\prime}, x', C'\right).$$

Consider a cost sharing problem (N', x', C') where $|N'| = 2$. Since ϕ satisfies ST, $\psi_i(N', q', c')$

$$= C'\left(\left(q_i' + x_i^{0\prime}, 0\right)\right) + \frac{1}{2}\left(C'\left(x'\right) - C'\left(\left(q_i' + x_i^{0\prime}, x_j^{0\prime}\right)\right) - C'\left(\left(x_i^{0\prime}, q_j' + x_j^{0\prime}\right)\right)\right)$$

for all $i \in N'$. Thus, $\psi(N', q', c') = Sh(N', v')$ where v' is defined as $v'(\{i\}) = C'\left(\left(x_i^{0\prime} + q_i', x_j^{0\prime}\right)\right)$ for all $i \in N'$ and $v'(N') = C'(x')$. This means that ψ coincides with the Shapley-Shubik rule for cost sharing problems with two agents. SI, AN and $CONS$ have been defined in cost sharing problems. The formal definition of these properties can be found, for instance, in Moulin (2002). Since ϕ satisfies SI, AN, and $CONS$ in PERT problems with delays, it is not difficult to check that ψ satisfies SI, AN and $CONS$ in cost sharing problems. By Theorem 5 in Friedman (2004), we conclude that ψ coincides with the Shapley-Shubik rule. Thus, for each cost sharing problem (N', q', c'),

$$\psi\left(N', q', c'\right) = Sh\left(N', v^{(x', C')}\right)$$

where $v^{(x',C')}(S) = C'\left(x^{0\prime}/x'_S\right)$. It is trivial to see that $v^{(x',C')}$ coincides with $v^{P'}$ where $P' = \left(N', \prec', x^{0\prime}, x', C'\right)$. Since $\prec'_N = \prec$, we deduce that $\phi\left(N, \prec, x^0, x, C\right) = \psi(N, x, C) = Sh\left(N, v^P\right)$. ■

An immediate consequence of Theorems 1 and 2 is that the Shapley rule is the unique rule satisfying AN, $CONS$, SI, and ST.

4 Other rules

Friedman and Moulin (1999) introduce the serial cost sharing rule for cost sharing problems, which is an extension of the serial cost sharing rule introduced by Moulin and Shenker (1992) in serial cost sharing problems. We define it in PERT problems with delays.

Given $t \in \mathbb{R}_+$ we denote $t_N = (t, ..., t) \in \mathbb{R}_+^N$. Given $x, y \in \mathbb{R}_+^N$, $(x \wedge y)$ is the vector of $\mathbb{R}_+^N$ where $(x \wedge y)_i = \min \{x_i, y_i\}$ for all $i \in N$.

For each $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$, the *serial rule* σ is defined as

$$\sigma_i (P) = \int_0^{x_i - x_i^0} \partial_i C \left(\left(x^0 + t_N\right) \wedge x\right) dt$$

for all $i \in N$.

Analogously, the *Aumann-Shapley rule* α can be defined in PERT problems with delays as

$$\alpha_i (P) = \left(x_i - x_i^0\right) \int_0^1 \partial_i C \left(x_0 + t \left(x - x^0\right)\right) dt$$

for all $i \in N$.

It is not difficult to check that both σ and α are well defined, *i.e.* they both satisfy EF and ND. Moreover, they both satisfy AD and DU.

However, we consider that σ and α are not suitable rules for PERT problems with delays. We clarify it in Example 3.

Example 3 *Consider a problem with two activities, say* 1 *and* 2*, which can be done simultaneously. The expected completion time of the activities are* 10 *and* 4*, respectively. The deadline of the project is 10. We must pay a cost of* 100 *for each unit of time the project is delayed. Assume that both activities employ* 12*. The delay of the project is* 2*, which means that the cost is* 200*. We believe that both activities have the same responsibility and, hence, both should pay 100. This situation can be modeled as a parallel PERT problem with delays* $P = (N, \prec, x^0, x, C)$ *where* $N = \{1, 2\}$*,* $\prec = \emptyset$*,* $x^0 = (10, 4)$*,* $x = (12, 12)$*, and* $C(x) = 100 (\max \{x_1, x_2\} - 10)$*. Making some computations we obtain that* $\sigma(P) = \alpha(P) = (200, 0)$*, which seems rather unfair.*

Through the paper we assumed that a rule should satisfy EF, ND, AD, and DU. EF and ND are fundamental properties in PERT problems with delays, like in cost sharing problems.

Nevertheless, in cost sharing problems there are reasonable rules that fail AD and DU. We mention two: the *axial rule* (Sprumont, 1998) and the *average cost pricing rule* (Moulin and Shenker, 1994).

The axial rule requires the cost function to be strictly increasing. Since this is not the usual case in PERT problems with delays, we conclude that the axial rule can not be computed for these problems.

Given $P = \left(N, \prec, x^0, x, C\right) \in \mathcal{P}$, the average cost pricing rule γ is defined as

$$\gamma_i\left(P\right) = \frac{x_i - x_i^0}{\sum_{j \in N}\left(x_i - x_i^0\right)} C\left(x\right)$$

for all $i \in N$.

Example 4 *Consider a problem with two activities, say* 1 *and* 2*, which can be done simultaneously. The expected completion time of the activities are* 20 *and* 4 *respectively. The deadline of the project is* 20*. We must pay* 100 *for each unit of time the project is delayed. Assume that activity* 1 *employs* 21 *and activity* 2 *employs* 13*. The delay of the project is one, which means that the cost is* 100*. Notice that the project is delayed only because activity* 1 *is delayed. We believe that all the cost should be paid by activity* 1*. This situation can be modeled by a parallel PERT problem with delays* $P = \left(N, \prec, x^0, x, C\right)$ *where* $N = \{1, 2\}$*,* $\prec = \emptyset$*,* $x^0 = (20, 4)$*,* $x = (21, 13)$*, and* $C\left(x\right) = 100\left(\max\left\{x_1, x_2\right\} - 20\right)$*. Thus,* $\gamma = (10, 90)$*, which seems rather unfair.*

Example 4 also shows that γ does not satisfy IID.

Brânzei et al. (2002) study PERT problems with delays where the cost function depends on the total delay of the project. They model it as a bankruptcy problem where the agents are the activities, the claim of each activity is its delay and the estate is the total delay of the project. They study, among others, the proportional rule, which coincides with γ. Notice that our criticism to γ also holds for any rule arising from bankruptcy problems.

References

Bergantiños, G. and Sánchez, E. (2002). How to distribute costs associated with a delayed project. *Annals of Operations Research*, 109(1-4):159–174.

Bergantiños, G. and Vidal-Puga, J. (2009). A value for PERT problems. *International Game Theory Review*, 11(04):419–436.

Brânzei, R., Ferrari, G., Fragnelli, V., and Tijs, S. (2002). Two approaches to the problem of sharing delay costs in joint projects. *Annals of Operations Research*, 109(1-4):359–374.

Friedman, E. and Moulin, H. (1999). Three methods to share joint costs or surplus. *Journal of Economic Theory*, 87(2):275 – 312.

Friedman, E. J. (2004). Paths and consistency in additive cost sharing. *International Journal of Game Theory*, 32(4):501–518.

Hart, S. and Mas-Colell, A. (1989). Potential, value, and consistency. *Econometrica*, 57(3):pp. 589–614.

Littlechild, S. and Owen, G. (1973). A simple expression for the Shapley value in a special case. *Management Science*, 20(3):370–372.

Moulin, H. (2002). Axiomatic cost and surplus sharing. In Arrow, K., Sen, A., and Suzumura, K., editors, *Handbook of Social Choice and Welfare*, volume I, chapter 6, pages 289–357. North-Holland, Amsterdam.

Moulin, H. and Shenker, S. (1992). Serial cost sharing. *Econometrica*, 60(5):pp. 1009–1037.

Moulin, H. and Shenker, S. (1994). Average cost pricing versus serial cost sharing: An axiomatic comparison. *Journal of Economic Theory*, 64(1):178–201.

Shapley, L. (1953). A value for n-person games. In Kuhn, H. and Tucker, A., editors, *Contributions to the theory of games*, volume II of *Annals of Mathematics Studies*, pages 307–317. Princeton University Press, Princeton NJ.

Shapley, L. and Shubik, M. (1954). A method for evaluating the distribution of power in a committee system. *American Political Science Review*, 48(3):787–792.

Shubik, M. (1962). Incentives, decentralized control, the assignment of joint costs and internal pricing. *Management Science*, 8(3):325–343.

Sprumont, Y. (1998). Ordinal cost sharing. *Journal of Economic Theory*, 81(1):126 – 162.

Thomson, W. (2009). Consistent allocation rules. Mimeo, University of Rochester.

Winter, E. (2002). The Shapley value. In Aumann, R. and Hart, S., editors, *Handbook of game theory with economic applications*, volume III of *Handbooks in Economics*. Elsevier Science, Amsterdam.