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Cooperative and axiomatic approaches to the knapsack allocation problem *

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Abstract

In the knapsack problem a group of agents want to fill a knapsack with several goods. Two issues should be considered. Firstly, to decide optimally the goods selected for the knapsack, which has been studied in many papers. Secondly, to divide the total revenue among the agents, which has been studied in few papers (including this one). We associate to each knapsack problem a cooperative game and we prove that the core is non-empty. Later, we follow the axiomatic approach. We propose two rules. The first one is based in the optimal solution of the knapsack problem. The second one is the Shapley value of the so called optimistic game. We offer axiomatic characterizations of both rules.

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1 Introduction

A mountaineer is planning to do a mountain tour with a knapsack, which has a limited size. Thus, he should decide which things should take with him in the knapsack. The idea is to select, given the limited size, the most important things. This is the so called knapsack problem. It has been applied in different real-world decisions. Some examples, see Pisinger and Toth (1998), are investments (we should decide how to invest a fixed amount of money between several business projects) and cargo airlines (we should decide how to fulfill an airplane given the demand of the costumers). Other examples, see Bretthauer and Setty (2002), include financial models, production and inventory management, stratified sampling, the optimal design of queueing network models in manufacturing, computer systems, and health care.

The most popular formulation is the so called 0-1 knapsack problem. There is a finite number of goods (one unit of each good) and we must decide which ones are selected for the knapsack. The goods can enter completely (1) or not enter at all (0). Since the number of goods is finite, there is an optimal solution. The first issue addressed is the computation of the optimal solution. Unfortunately, this problem is NP complete. Thus, we should define algorithms for approximating the optimal solution.

There are more general formulations of the knapsack problem. We mention some of them. The continuous knapsack problem, where we can include in the knapsack fractions of each good. The bounded knapsack problem where we can have several copies of each good. The d-dimensional knapsack problem where we have several constraints (for instance weight and volume) for fulfill the knapsack. The multiple knapsack problem where we have several knapsacks instead of only one. The multiple choice knapsack problem where there are several types of objects and we must select one object of each type. The non-linear knapsack problem where the objective function and the constraint are non-linear. Again, the main issued addressed by this literature is how to compute the optimal solution. Pisinger and Toth (1998), Martello *et al* (2000), and Kellerer *et al* (2004) give reviews of this literature.

In many cases the computation of the optimal solution (or the approximation obtained) is only the first part of the problem. The second part is to divide the cost (or benefits) among the agents. Whereas the first part is mainly studied in the Operations Research literature, the second part is studied also in Economics. For instance, in the minimum cost spanning tree problem, Bird (1976), Kar (2002), Dutta and Kar (2004), Tijs *et al* (2006), Bergantiños and Vidal-Puga (2007a), Bogomolnaia and Moulin (2010), and Trudeau (2012) propose several rules for allocating the cost of the optimal solution among the agents. Borm *et al* (2001) give a survey of this literature focusing on connection problems, routing (Chinese postman and travelling salesman), scheduling (sequencing, permutation, assignment), production (linear production, flow) and inventory.

As far as we know, Darmann and Klamler (2014) is the only paper in which this second part is studied in the knapsack problem. They focus in the continuous knapsack problem, where the optimal solution could be computed in polynomial time. They consider the following "the goal is to divide the cost of the optimally packed knapsack among the individuals in a fair manner. In this paper, we assume that every unit of weight imposes a cost of one, and therefore the total cost of the knapsack is equal to the weight constraint W". Then, they define a family of rules, which is characterized with several properties. Besides, they characterize the rule of the family that divides the cost associated with each good equally among the agents approving such good.

Even our paper also consider the second part of the problem, our approach is different. Darmann and Klamler (2014) consider the case where agents either approve or disapprove each good. We consider a more general case where each agent i has a utility p_j^i for each unit of good j included in the knapsack. Darmann and Klamler (2014) corresponds to the case where for each i and j, $p_j^i = 1$ when agent i approves good j and $p_j^i = 0$ when agent i disapproves good j. Moreover, our goal is to divide the total utility generated by the optimal knapsack among the agents.

Let us clarify the difference between both approaches with a trivial example. Consider the knapsack problem with three agents (1, 2, and 3) and two goods (a and b). The size of the knapsack is 1 and the size of each good is also 1. Good a is approved by agents 1 and 2 and good b is approved by agent 3. In our model $p_a^1 = p_a^2 = p_b^3 = 1$ and $p_b^1 = p_b^2 = p_a^3 = 0$. To include good a in the knapsack generates a profit of 2 (agent 1 and 2 generates a profit of 1 and agent 3 generates 0). To include good b generates a profit of 1 (agent 1 and 2 generates a profit of 0 and agent 3 generates 1). The optimal solution is to include good a in the knapsack. In Darmann and Klamler (2014) agents 1 and 2 pays 0.5 and agent 3 pays nothing. This means that agent 1 and 2 obtain some earnings (the utility they get from good a minus the amount they pay) whereas agent 3 obtains nothing (he receives nothing and he pays nothing). In our case agents must decide how to divide the utility generated by the optimal solution (2 in this case) among the agents. Thus, we also consider the possibility that agent 3 is compensated by agents 1 and 2 (because good b is not included) and agent 3 obtains some profit. Actually, one of the allocation we consider, do it.

We assume that a group of agents (N) should decide which goods (from a set M) to include in a knapsack of fixed size W. Each good $j \in M$ has a fixed size w_j. Preferences of the agents over the goods are heterogeneous and modelled by a vector p where for each $i \in N$ and $j \in M$, $p_j^i \in \mathbb{R}_+$ denotes the utility obtained by agent i when a unit of good j is included in the knapsack. We assume that agents will select the goods maximizing the total utility. In this paper we follow a cooperative approach and we study how to divide the total utility among the agents. Thus, we implicitly assume that agents that include in the knapsack many of "their goods" could compensate to those agents who include few of "their goods" in order to obtain a more fair allocation.

In the literature there is a way for associating to each knapsack problem a cooperative game (see, for instance Kellerer *et al* (2004)). The value of a coalition S is defined as the utility obtained by agents of this coalition when the knapsack is fulfilled in the worst way for S. We call this game the pessimistic game. It is known that the core of this game is non empty and contains the allocation induced by the optimal solution. We introduce two alternative cooperative games; the optimistic game and the realistic game. In the optimistic game the value of a coalition S is defined as the utility obtained by agents of this coalition when the knapsack is fulfilled in the best way for S. It is easy to see that the core of the optimistic game could be empty. In the realistic game the value of a coalition S is defined as the utility obtained by agents of this coalition when agents in $N \backslash S$ fulfill the knapsack in the best way for $N \backslash S$. We prove that the realistic game has a non empty core containing the allocation induced by the optimal solution.

Later on we follow the axiomatic approach. A knapsack rule is a function that for each knapsack problem selects the goods we include in the knapsack, and the way in which the total utility generated these goods is divided among the agents. We introduce several properties of rules and we discuss some relationships between the properties. One of them is core selection, which says that the allocation should be in the core of the realistic game. Core selection implies, in several knapsack problems, that some agents could receive 0, which seems a little bit unfair. Thus, we also consider the securement property (inspired in Moreno-Ternero and Villar (2004)), which guarantees to any agent a minimal amount. Securement says that each agent should receive, at least, $\frac{1}{n}$ the amount he will obtain when the knapsack is assigned to him. Unfortunately there is no rule satisfying both properties. Thus, we consider two rules one satisfying each of the properties.

We first consider the rule induced by the optimal solution. This rule allocates to each agent the utility obtained by this agent under the optimal solution. It satisfies core selection and then fails securement. We present three characterizations of this rule. In the first one we use core selection and no advantageous splitting. In the second one we use efficiency, maximum aspirations, independence of irrelevant goods and composition up. In the third one we use efficiency, maximum aspirations and no advantageous splitting.

We later consider the Shapley value of the optimistic game, which satisfies securement but fails core selection. We characterize it with efficiency and equal contributions.

The rest of the paper is organized as follows. In Section 2 we formally introduce the knapsack problem. In Section 3 we study the three cooperative games associated with the knapsack problem. In Section 4 we introduce the properties, the rules, and the axiomatic characterizations. In Appendix we present some omitted proofs of our results. Finally, we give the list of the references.

2 The knapsack problem

In the knapsack problem a set of agents (N) want to include some goods (M) in a knapsack of size W.

We assume that the set of potential agents is infinite. Then, there exists an infinite set $\mathcal{N}$ such that $N \subset \mathcal{N}$.

We focus in the continuous knapsack problem, where it is assumed that goods are perfectly divisible. Then we can select fractions of each good to be included in the knapsack.

A **knapsack problem** is defined as a 5-tuple $P = (N, M, W, w, p)$ where

- $N = \{1, ..., n\}$ denotes a set of agents.
- $M = \{g_1, ..., g_m\}$ denotes the set of goods.

- $W \in \mathbb{R}_+$ is the size of the knapsack.
- $w = \{w_j\}_{j \in M}$ where for each $j \in M$, w_j denotes the size of good j.
- $p = \{p_j^i\}_{i \in N, j \in M}$ where for each $i \in N$, $j \in M$, $p_j^i \in \mathbb{R}_+$ denotes the utility that agent i obtains for each unit of good j that is included in the knapsack.

Darmann and Klamler (2014) consider the case where $p_j^i \in \{0, 1\}$ for each $i \in N$, $j \in M$. Namely, agents approve or disapprove each good. Thus, our model is more general.

We introduce some notation used later.

Given $S \subset N$, let P^S denote the knapsack problem induced by P when the set of agents is S. Namely $P^S = (S, M, W, w, p^S)$ where $p^S = \{p_j^i\}_{i \in S, j \in M}$.

Given $T \subset M$, let P^T denote the knapsack problem induced by P when the set of goods is T. Namely $P^T = \left(N, T, W, (w_j)_{j \in T}, p^T\right)$ where $p^T = \{p_j^i\}_{i \in N, j \in T}$.

For each $j \in M$,

$$p_j = \sum_{i \in N} p_j^i \tag{1}$$

is a measure of the importance of good j for the set of agents.

Besides, for each $S \subset N$ and $j \in M$, $p_j^S = \sum_{i \in S} p_j^i$. Notice that for each $j \in M$, $p_j^N = p_j$.

We say that $x = (x_j)_{j \in M} \in \mathbb{R}^M$ is a **feasible solution** for P if $x_j \in [0, 1]$ for each $j \in M$ and $\sum_{j \in M} w_j x_j = W$. For each $j \in M$, $w_j x_j$ is the space occupied by good j in the knapsack. We denote by $FS(P)$ the set of feasible solutions for P.

The interesting case arises when we can not include in the knapsack all goods, namely, $W < \sum_{j \in M} w_j$. In this case $FS(P)$ has many points. The case $W \geq \sum_{j \in M} w_j$ is solved easily by including all goods in the knapsack. That is $x_j = 1$ for all $j \in M$.

Each feasible solution x induces a vector of utilities $u(x) = (u_i(x))_{i \in N}$ given by the goods we have included in the knapsack. For each feasible solution x and each $i \in N$,

$$u_i(x) = \sum_{j \in M} p_j^i x_j.$$

The first question addressed in the literature (mainly from Operations Research) is to select the goods to be included in the knapsack in such a way that the sum of the utilities of the agents is maximized. Formally,

$$\max_{x \in FS(P)} \sum_{i \in N} u_i(x). \tag{2}$$

In what follows, we assume, without loss of generality, that the goods are sorted in such a way that

$$\frac{p_1}{w_1} \geq ... \geq \frac{p_m}{w_m}.$$

This problem has at least one **optimal solution**. One of them is $x^*(P) = \left\{x_j^*(P)\right\}_{j \in M}$ defined as

$$x_j^*(P) := \begin{cases} 1 \text{ if } j = 1, ..., s-1 \\ \dfrac{1}{w_s}\left(W - \displaystyle\sum_{k=1}^{s-1} w_k\right) \text{ if } j = s \\ 0 \text{ if } j = s+1, ...m \end{cases} \tag{3}$$

where s is defined by

$$\sum_{k=1}^{s-1} w_k < W \leq \sum_{k=1}^{s} w_k.$$

When no confusion arises we write x^* instead of $x^*(P)$. We will denote by $X^*(P)$ (or X^*) the set of all optimal solutions in P.

If we assume that $\frac{p_1}{w_1} > ... > \frac{p_m}{w_m}$, we can guarantee that the previous problem has a unique optimal solution.

We denote by $\mathcal{P}$ the class of all knapsack problems and by $\mathcal{P}^*$ the class of knapsack problems where $\frac{p_1}{w_1} > ... > \frac{p_m}{w_m}$.

We assume that agents decide the goods to be included in the knapsack. They also decide the way in which the total utility generated by the selected goods is divided among them.

For any problem P the set of feasible allocations is defined as

$$FA(P) = \left\{ (y_i)_{i \in N} \in \mathbb{R}^N_+ : \sum_{i \in N} y_i = \sum_{i \in N} u_i(x) \text{ for some } x \in FS(P) \right\}.$$

3 Knapsack cooperative games

In this section we associate with each knapsack problem the cooperative games with transferable utility called: pessimistic, optimistic and realistic. We study the core of such games. The core of the pessimistic and realistic game are always non empty whereas the core of the optimistic game could be empty.

A **cooperative game with transferable utility** (briefly, a TU game) is a pair (N, v) where $v : 2^N \to \mathbb{R}$ satisfies $v(\varnothing) = 0$. When no confusion arises we write v instead of (N, v).

The **core** of a TU game (N, v) is defined as

$$c(v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N) \text{ and for each } S \subset N, \sum_{i \in S} x_i \geq v(S) \right\}.$$

We first associate with each problem P several cooperative games, depending on the way we define the value of a coalition S.

In the **pessimistic** approach we assume that the knapsack is fulfilled in the worst way for S. Formally, for each knapsack problem P we define the game (N, v^p_P) where for each $S \subset N$,

$$v^p_P(S) = \min_{x \in FS(P)} \sum_{i \in S} u_i(x).$$

When no confusion arises we write v^p instead of v^p_P.

In the **optimistic** approach we assume that agents in S can fulfill the knapsack the way they want. Formally, for each knapsack problem P we define the game (N, v^o_P) where for each $S \subset N$,

$$v^o_P(S) = \max_{x \in FS(P)} \sum_{i \in S} u_i(x).$$

When no confusion arises we write v^o instead of v^o_P.

In the **realistic** approach we assume that agents in $N \backslash S$ fulfill the knapsack in the best way for them and then agents in S optimize the space of the knapsack left by N/S. Let $X^{*N \backslash S}$ the set of optimal solutions of the problem $P^{N \backslash S}$. For each knapsack problem P we define the game (N, v^r_P) where for each $S \subset N$,

$$v^r_P(S) = \max_{x \in X^{*N \backslash S}} \sum_{i \in S} u_i(x).$$

When no confusion arises we write v^r instead of v^r_P.

Remark 1 *It is obvious that for each problem P and each $S \subset N$, $v^p(S) \leq v^r(S) \leq v^o(S)$ and $v^p(N) = v^r(N) = v^o(N)$. Then,*

$$c(N, v^o) \subset c(N, v^r) \subset c(N, v^p).$$

Example 1 *Let P be such that $N = \{1,2,3\}$, $M = \{a,b,c\}$, $W = 2$ and $w_j = 1$ for all $j \in M$. Besides the vector p satisfies the following conditions.*

- *Agent 1 is interested in good a but not in the others. Namely, $p^1_a > 0$ and $p^1_j = 0$ otherwise.*
- *Agents 2 and 3 prefer b to c and they are not interested in good a. Furthermore, they prefer good c more that agent 1 prefer good a. Namely, for each agent $i \neq 1$, $p^i_b > p^i_c > p^1_a$ and $p^i_a = 0$.*
- *Agent 2 is more interested in objects of $M \backslash \{a\}$ than agent 3. Namely, $p^2_j > p^3_j$, for each $j \in \{b,c\}$.*

We now compute the three games. We detail the computation for coalition $\{2,3\}$. *The worst feasible solution for agents* 2 *and* 3 *is to include goods* a *and* c. *Thus,* $v^p(2,3) = p_c^2 + p_c^3$. *The best feasible solutions for agent are* $\{a\}$, $\{a,b\}$, *and* $\{a,c\}$. *Among them agents* 2 *and* 3 *prefer* $\{a,b\}$. *Then,* $v^r(2,3) = p_b^2 + p_b^3$. *The best feasible solution for agents* 2 *and* 3 *is to select goods* b *and* c. *Then,* $v^o(2,3) = p_b^2 + p_b^3 + p_c^2 + p_c^3$.

T	$v^p(T)$	$v^r(T)$	$v^o(T)$
$\{1\}$	0	0	p_a^1
$\{2\}$	p_c^2	$p_b^2 + p_c^2$	$p_b^2 + p_c^2$
$\{3\}$	p_c^3	$p_b^3 + p_c^3$	$p_b^3 + p_c^3$
$\{1,2\}$	$p_a^1 + p_c^2$	$p_b^2 + p_c^2$	$p_b^2 + p_c^2$
$\{1,3\}$	$p_a^1 + p_c^3$	$p_b^3 + p_c^3$	$p_b^3 + p_c^3$
$\{2,3\}$	$p_c^2 + p_c^3$	$p_b^2 + p_b^3$	$p_b^2 + p_b^3 + p_c^2 + p_c^3$
N	$p_b^2 + p_b^3 + p_c^2 + p_c^3$	$p_b^2 + p_b^3 + p_c^2 + p_c^3$	$p_b^2 + p_b^3 + p_c^2 + p_c^3$

The core of the pessimistic game v^p is non empty and contains $u(x^*)$ for all $x^* \in X^*$ (see, for instance, Kellerer *et al*. (2004)).

The core of the optimistic game v^o could be empty as next example shows.

Example 2 *Let* P *be such that* $N = \{1,2\}$, $M = \{a,b\}$, $W = 1$, $w_a = w_b = 1$, $p_a^1 = 1$, $p_b^2 = 0.9$ *and* $p_b^1 = p_a^2 = 0$. *Now* $v^o(1) = 1$, $v^o(2) = 0.9$, *and* $v^o(1,2) = 1$. *Thus,* $c(v^o) = \varnothing$.

We now prove that the core of the realistic game v^r is non-empty because $u(x^*)$ belongs to such core.

Theorem 1 *For each knapsack problem* P, $u(x') \in c(v^r)$ *for all* $x' \in X^*$.

Proof. Let P be a problem and $x' \in X^*$ such that $u(x') \notin c(v^r)$. Then, it exists $S \subset N$ such that $v^r(S) > \sum_{j \in M} p_j^S x_j'$.

Let $x \in X^{*N \setminus S}$ be such that

$$v^r(S) = \sum_{j \in M} p_j^S x_j > \sum_{j \in M} p_j^S x_j'$$

As $x \in X^{*N \setminus S}$, $\sum_{j \in M} p_j^{N \setminus S} x_j \geq \sum_{j \in M} p_j^{N/S} x_j'$. Then,

$$\begin{aligned} \sum_{i \in N} u_i(x) &= \sum_{i \in N} \sum_{j \in M} p_j^i x_j = \sum_{j \in M} p_j^S x_j + \sum_{j \in M} p_j^{N/S} x_j > \sum_{j \in M} p_j^S x_j' + \sum_{j \in M} p_j^{N/S} x_j' \\ &= \sum_{j \in M} p_j x_j' = \max_{x \in S(P)} \sum_{i \in N} u_i(x), \end{aligned}$$

which is a contradiction. ■

The next example shows that the core of v^r could have more points.

Example 3 *Let P be such that $N = \{1, 2, 3\}$, $M = \{a, b, c, d\}$, $W = 5$, $w_a = w_b = w_d = 2$, $w_c = 1$, $p_d^1 = 0.7$, $p_a^1 = p_b^1 = p_c^1 = 0$, $p_a^2 = p_b^2 = 1$, $p_c^2 = p_d^2 = 0$, $p_a^3 = 1, p_b^3 = 0.9$, $p_c^3 = 0.8$, and $p_d^3 = 0$. Then, $p_a = 2$, $p_b = 1.9$, $p_c = 0.8$, $p_d = 0.7$,*

$$\frac{p_a}{w_a} = \frac{2}{2} > \frac{p_b}{w_b} = \frac{1.9}{2} > \frac{p_c}{w_c} = \frac{0.8}{1} > \frac{p_d}{w_d} = \frac{0.7}{2}.$$

The optimal solution is $x^* = (1, 1, 1, 0)$. Namely, we include in the knapsack a, b and c. Now $u(x^*) = (0, 2, 2.7)$.

$v^r(1) = 0$, $v^r(2) = 2$, $v^r(3) = 1.9$, $v^r(1, 2) = 2$, $v^r(1, 3) = 2.7$, $v^r(2, 3) = 3.45$, and $v^r(N) = 4.7$. Then v^r has many core elements different from $u(x^*)$. For instance, $(0.7, 2, 2)$.

4 Knapsack rules and properties

In this section we introduce several properties of rules. We discuss some relationships between the properties. Core selection says that we must select an allocation in the realistic core. Rules selecting allocations in the core could be unfair because agents who want not very demanded goods (those with $\frac{p_j}{w_j}$ small) could receive zero. Thus, we consider the property of securement, which says that each agents must receive a minimum amount. Unfortunately no rule satisfy both properties.

We then introduce two rules. The first one, based in the optimal solution, satisfies core selection. The second one, based in the Shapley value, satisfies securement. We study the properties satisfied by each rule. Besides we provide several axiomatic characterizations of both rules.

A **rule** is a function f assigning to each problem P a pair $f(P) = (f^1(P), f^2(P))$ where $f^1(P) \in FS(P)$ and $\sum_{i \in N} f_i^2(P) = \sum_{i \in N} u_i(f^1(P))$. Notice that $f^1(P)$ denote the goods we include in the knapsack and $f^2(P)$ denotes the way in which the total utility generated by $f^1(P)$ is divided among the agents.

We now introduce several properties of rules and we discuss some relationships between the properties.

Efficiency says that $f^2(P)$ the vector of allocations proposed by f is not Pareto dominated in the set of feasible allocations $FA(P)$.

Efficiency (ef)**.** For each problem P, $\sum_{i \in N} f_i^2(P) = \max_{x \in FS(P)} \sum_{i \in N} u_i(x)$.

In $\mathcal{P}$ efficiency says that $f^1(P) \in X^*$. In $\mathcal{P}^*$ efficiency means that $f^1(P) = x^*$.

Symmetry says that if two agents give the same utility to each good, then both receive the same allocation.

Symmetry (sym)**.** For each problem P and each $i, i' \in N$ such that $p^i = p^{i'}$, then $f_i^2(P) = f_{i'}^2(P)$.

If the valuation of agent i to some goods increases, then the allocation to agent i can not decrease.

Monotonicity (mon)**.** Consider two problems $P = (N, M, W, w, p)$ and $P' = (N, M, W, w, p')$ such that there exists $i \in N$ satisfying $p_j'^i \geq p_j^i$ for all $j \in M$ and $p_j'^k = p_j^k$ for all $j \in M$ and $k \in N \setminus \{i\}$. Then, $f_i^2(P') \geq f_i^2(P)$.

Dummy says that if some agent is not interested in any good, then he receives nothing.

Dummy (dum)**.** For each problem P and each $i \in N$ such that $p_j^i = 0$ for each $j \in M$, then $f_i^2(P) = 0$.

Core selection says that the allocation proposed by the rule $(f^2(P))$ should belong to the core of the problem. In this case, we select the core of the realistic game because we find it the more suitable for this class of problems.

Core selection (cs). For each problem P, $f^2(P) \in c(v^r)$.

It is clear that core selections implies efficiency.

Assume that we remove a good not selected by the optimal solution, then the rule does not change. This property is inspired in the well known principle of independence of irrelevant alternatives (used, for instance, in bargaining problems by Nash (1950)).

Independence of irrelevant goods (iig)**.** Let P be a problem and $j \in M$ satisfying that $x_j = 0$ for any optimal solution x. Then, $f(P) = f\left(P^{M \setminus \{j\}}\right)$.

Composition up says that we can fulfill the knapsack in one step or, first fulfill some part of the knapsack and later the remaining. This property has been used in several economics problems. See for instance the surveys of Thomson (2003, 2015) about bankruptcy problems. Darmann and Klamler (2014) also uses this property.

For each problem $P = (N, M, W, w, p)$, $W_1 \leq W$ and $x \in [0,1]^M$ we define the problems

$$\begin{aligned} P(W_1) &= (N, M, W_1, w, p) \text{ and} \\ P(W - W_1, x) &= (N, M_x, W - W_1, w_x, p_x) \end{aligned}$$

where

$$\begin{aligned} M_x &= \{j \in M : x_j < 1\}, \\ (w_x)_j &= w_j(1 - x_j) \text{ for each } j \in M_x, \text{ and} \\ (p_x)_j^i &= p_j^i \text{ for each } i \in N \text{ and } j \in M_x. \end{aligned}$$

Composition up (cu)**.** For each problem P and each $W_1 \leq W$,

$$\begin{aligned} f_j^1(P) &= f_j^1(P(W_1)) + f_j^1\left(P\left(W - W_1, f^1(P(W_1))\right)\right) \text{ for all } j \in M \text{ and} \\ f_i^2(P) &= f_i^2(P(W_1)) + f_i^2\left(P\left(W - W_1, f^1(P(W_1))\right)\right) \text{ for all } i \in N. \end{aligned}$$

We now introduce two properties closely related. Actually in several papers both properties appear as a single one. No advantageous merging means that no group of agents has incentives to pool their utility and to present themselves as a single agent. No advantageous splitting means that no agent has incentives to divide his utility and to present himself as a group of agents.

No advantageous merging (nam). For each problems $P = (N, M, W, w, p)$ and $P' = (N', M, W, w, p')$ where $N \subset N'$ and there exists $i \in N$ such that $p^i = p'^i + \sum_{k \in N' \backslash N} p'^k$ and $p^k = p'^k$ for all $k \in N' \backslash N$,

$$f_i^2(P') + \sum_{k \in N' \backslash N} f_k^2(P') \geq f_i^2(P).$$

No advantageous splitting (nas). For each problems $P = (N, M, W, w, p)$ and $P' = (N', M, W, w, p')$ where $N \subset N'$ and there exists $i \in N$, $p^i = p'^i + \sum_{k \in N' \backslash N} p'^k$ and $p^k = p'^k$ for all $t \in N' \backslash N$,

$$f_i^2(P') + \sum_{k \in N' \backslash N} f_k^2(P') \leq f_i^2(P).$$

Darmann and Klamler (2014) consider the property of pairwise merge-and-split-proofness which is related, in the motivation, with nam and nas. Both properties are inspired in the property of strategy-proof introduced in O'Neill (1982). Actually we define it in the same way as they appear in Thomson (2003, 2015). There are two differences between pairwise merge-and-split-proofness and $nam+nas$. First, when an agent is divided in several ones (or several ones join in a single agent), in Darmann and Klamler (2014) each agent must approve different goods. Since our model is more general we allow that different agents approve the same good. Second, in Darmann and Klamler (2014) the property says that agents who do not merge or split should not be affected. In our case (as in the bankruptcy problem) we say that agents that merge or split do not improve.

The idea of the following property is to give an upper bound to the amount received by each agent. In our case, each agent could receive, at most, the amount he will receive when he can use the whole knapsack.

For each problem P and each $i \in N$ we define the **maximum aspiration** of agent i as $MA_i(P) = \max_{x \in FS(P)} u_i(x)$. Notice that $MA_i(P) = v^o(i)$.

Maximum aspirations (ma). For each problem P and each $i \in N$, $f_i^2(P) \leq MA_i(P)$.

The idea of the following property is dual of the previous one, is to guarantee to each agent a minimum amount. In our case, each agents should receive at least $\frac{1}{n}$ the amount he will obtain when the knapsack is assigned to him. This property is inspired in the securement property introduced by Moreno-Ternero and Villar (2004) for bankruptcy problems.

For each problem P and each $i \in N$ we define the **secure allocation** of agent i as

$$SE_i(P) = \frac{1}{n} \max_{x \in FS(P)} u_i(x).$$

Notice that $SE_i(P) = \frac{v^o(i)}{n}$.

Securement (se). For each problem P and each $i \in N$, $f_i^2(P) \geq SE_i(P)$.

Equal contributions is a principle widely used in the literature since Myerson (1980) introduced it in TU games. It says that if agent i leaves the problem the change in the allocation of agent j coincides with the change in the allocation to agent j when agent i leaves the problem.

Equal contributions (ec). For each problem P and each $i, k \in N$,

$$f_i^2(P) - f_i^2\left(P^{N\setminus\{k\}}\right) = f_k^2(P) - f_k^2\left(P^{N\setminus\{i\}}\right).$$

All the previous properties can be considered desirable for a rule, but clearly we could have incompatibility between them. For example, if we restrict our attention to rules satisfying core selection (securement) we must leave off securement (core selection) because both properties are incompatible. We also prove that, under dummy and efficiency, independence if irrelevant goods and securement are incompatible. In the next proposition we study these relations between the properties.

Proposition 1 (1) *There is no rule satisfying core selection and securement.*

(2) *Let f be a rule satisfying dummy and efficiency. Then, f does not satisfy independence of irrelevant goods and securement.*

Proof. (1) Let f be a rule satisfying cs and se. Consider Example 1. For each $i \in N\backslash\{1\}$, $v^r(i) = \sum_{j\in M} p_j^i x_j^* = u_i(x^*)$ because $x^{*N\backslash\{i\}} = x^* = (0,1,1)$. Now,

$$v^r(N) = \sum_{i\in N} u_i(x^*) = \sum_{i\in N\backslash\{1\}} \sum_{j\in M} p_j^i x_j^* = \sum_{i\in N\backslash\{1\}} v^r(i).$$

Then, $c(v^r) = (u_i(x^*))_{i\in N}$. Since f satisfies cs, $f_1^2(P) = u_1(x^*) = 0$. Since f satisfies se, $f_1^2(P) \geq SE_1(P) = \frac{p_a^1}{3}$, which is a contradiction.

(2) Consider Example 2. Then, $SE_1(P) = 0.5$, $SE_2(P) = 0.45$ and $x^* = (1,0)$. If f satisfies ef and iig, $f_2^2(P) = f_2^2\left(P^{\{a\}}\right)$. Since f satisfies dum, $f_2^2\left(P^{\{a\}}\right) = 0$. Then, $f_2^2(P) = 0$. If f satisfies se we have that $f_2(P) \geq 0.45$, which is a contradiction. ■

Core selection is a quite standard property in the literature. It would be nice for an allocation to be in the core. Nevertheless, the allocations in the core could be very unfair. In the knapsack problem it could also happen. For instance, in Example 1 we have only one core allocation, which gives 0 to agent 1. Thus, if we try to find a fair allocation sometimes is better to look outside the core. For instance, in TU games, the Shapley value, the most popular fair allocation, could be outside the core.

We think that securement is a nice fairness property because it guarantees that all non-dummy agents will receive something. For instance, in Example 1 it says that agent 1 will receive something.

By Proposition 1 core selection and securement are incompatible. Since we consider both properties very interesting, we will study two rules in the paper. One satisfying core selection and the other satisfying securement.

4.1 The rule induced by the optimal solution

In this section we study a rule satisfying core selection. We focus in the rule induced by the optimal solution to the knapsack problem. We fill the knapsack in the optimal way and each agent receives the utility given by the knapsack. Namely, there is no transfers among the agents. Since in a general knapsack problem we can have several optimal solutions, we restrict our study to $\mathcal{P}^*$, where the optimal solution is unique and then well defined. Notice that from an strict mathematical point of view our restriction is not important because the measure of $\mathcal{P}\backslash\mathcal{P}^*$ in $\mathcal{P}$ is zero. We study the properties satisfied by this rule and we give several axiomatic characterizations.

Given $P \in \mathcal{P}^*$, let x^* denote the unique optimal solution of P. Making an abuse of notation we denote the rule induced by x^* also as x^*. Namely, let x^* be the rule defined as $x^{*1}(P) = x^*$ and $x_i^{*2}(P) = u_i(x^*)$ for all $i \in N$.

The optimal solution has been used by Darmann and Klamler (2014) for defining a rule in his model. The cost associated with each good, selected by the optimal solution, is divided equally among the agents approving such good.

We now study the properties of rule x^*.

Proposition 2 (1) *The rule x^* satisfies efficiency, symmetry, monotonicity, dummy, core selection, independence of irrelevant goods, composition up, no advantageous merging, no advantageous splitting, and maximum aspirations.*

(2) *The rule x^* does not satisfy securement and equal contributions.*

The proof is in Appendix.

In the next theorem we give several axiomatic characterizations of the optimal rule.

Theorem 2 (1) *x^* is the unique rule satisfying core selection and no advantageous splitting.*

(2) *x^* is the unique rule satisfying efficiency, independence of irrelevant goods, composition up, and maximum aspirations.*

(3) *x^* is the unique rule satisfying efficiency, no advantageous splitting, and maximum aspirations.*

Besides, the properties used in the previous characterizations are independent.

The proof is in Appendix.

Remark 2 *If we check the proof of (1) in Theorem 2 we realize that we can replace core selection by efficiency and individual rationality (for each problem P, each agent $i \in N$ must receives at least $v_P^r(i)$).*

4.2 The rule induced by the Shapley value

In this section we study a rule satisfying securement. We fill the knapsack in the optimal way and each agent receives the utility given by Shapley value of the optimistic game associated with the knapsack problem[1]. In this section we consider the set of all problems $\mathcal{P}$. We study the properties satisfied by this rule and we give an axiomatic characterizations.

The **Shapley value** of a game (N, v) (Shapley, 1953) is denoted by $Sh(v)$. For each $i \in N$ we have that

$$Sh_i(v) = \sum_{S \subset N \setminus \{i\}} \frac{s!(n-s-1)!}{n!} (v(S \cup \{i\}) - v(S)).$$

Given $P \in \mathcal{P}$, let x^* denote an optimal solution of P. We define the **optimistic Shapley rule,** denoted by Sh^o, as the rule indued by the Shapley value of the optimistic game. Namely, $Sh^{o1}(P) = x^*$ and $Sh^{o2}(P) = Sh(v_P^o)$.

We now study the properties satisfied by the optimistic Shapley rule.

Proposition 3 (1) *The optimistic Shapley rule satisfies efficiency, symmetry, monotonicity, dummy, maximum aspirations, securement, and equal contributions.*

(2) *The optimistic Shapley rule does not satisfy core selection, independence of irrelevant goods, composition up, no advantageous merging and no advantageous splitting.*

The proof is in Appendix.

We now give a characterization of Sh^o.

Theorem 3 *The optimistic Shapley rule is the unique rule satisfying efficiency and equal contributions.*

Besides, the properties are independent.

The proof is in Appendix.

[1] There are other papers where the it is studied the Shapley value of the optimisitic game. For instance Bergantiños and Vidal-Puga (2007b) study it in minimum cost spanning tree problems.

5 Appendix: Proofs of the results

Proof of Proposition 2. (1) It is obvious that x^* satisfies ef, sym, mon, dum, iig, and ma.

We now prove that x^* satisfies cu. We know that there exists $s \in \mathbb{N}$ such that $x_j^{*1}(P) = 1$ for all $j < s$, $0 < x_s^{*1}(P) \leq 1$, and $x_j^{*1}(P) = 0$ for all $j > s$. Let P and $W_1 \leq W$. Then, it exists $t \leq s$ such that $x_j^{*1}(P(W_1)) = 1$ for all $j < t$, $0 < x_t^{*1}(P(W_1)) \leq 1$, and $x_j^{*1}(P(W_1)) = 0$ for all $j > t$.

Assume that $x_t^{*1}(P(W_1)) < 1$ and $t < s$ (the other cases are similar and we omit it). Then $M_{x^{*1}(P(W_1))} = \{t, ..., m\}$, $\left(w_{x^{*1}(P(W_1))}\right)_t < w_t$, $\left(w_{x^{*1}(P(W_1))}\right)_j = w_j$ for all $j > t$, and the values of p and $p_{x^{*1}(P(W_1))}$ coincide. Thus,

$$\begin{aligned} \frac{\left(p_{x^{*1}(P(W_1))}\right)_t}{\left(w_{x^{*1}(P(W_1))}\right)_t} &= \frac{p_t}{\left(w_{x^{*1}(P(W_1))}\right)_t} > \frac{p_t}{w_t} > \frac{p_{t-1}}{w_{t-1}} = \frac{\left(p_{x^{*1}(P(W_1))}\right)_{t-1}}{\left(w_{x^{*1}(P(W_1))}\right)_{t-1}} > \\ &> \ ... > \frac{p_m}{w_m} = \frac{\left(p_{x^{*1}(P(W_1))}\right)_m}{\left(w_{x^{*1}(P(W_1))}\right)_m}. \end{aligned}$$

Now it is obvious that $x_t^{*1}(P(W - W_1, x^{*1}(P(W_1)))) = w_t - \left(w_{x^{*1}(P(W_1))}\right)_t$ and $x_j^{*1}(P(W - W_1, x^{*1}(P(W_1)))) = x_j^{*1}(P(W_1))$ for all $j > t$. Then,

$$x_j^{*1}(P) = x_j^{*1}(P(W_1)) + x_j^{*1}\left(P\left(W - W_1, x^{*1}(P(W_1))\right)\right) \text{ for all } j \in M.$$

Because of the previous expression and the definition of f^2 it is obvious that

$$x_i^{*2}(P) = x_i^{*2}(P(W_1)) + x_i^{*2}\left(P\left(W - W_1, x^{*1}(P(W_1))\right)\right) \text{ for all } i \in N.$$

Then x^* satisfies cu.

By Proposition 1, x^* satisfies core selection.

Let P and P' be as in the definition of nam and nas. Since $p_j = p'_j$ for all $j \in M$, $x^{*1}(P) = x^{*1}(P')$ and

$$\begin{aligned} x_i^{*2}(P') + \sum_{k \in N' \setminus N} x_k^{*2}(P') &= \sum_{j \in M} p_j'^i x_j^{*1}(P') + \sum_{j \in M} \sum_{k \in N' \setminus N} p_j'^k x_j^{*1}(P') \\ &= \sum_{j \in M} \sum_{k \in N' \setminus N} (p_j'^i + p_j'^k) x_j^{*1}(P') \\ &= \sum_{j \in M} p_j^i x_j^{*1}(P) \\ &= x_i^{*2}(P). \end{aligned}$$

Thus, x^* satisfies *nam* and *nas*.

(2) Consider the problem given by Example 2. Thus, $x^{1*}(P) = (1,0)$, $x^{2*}(P) = (x_1^{*2}(P), x_1^{*2}(P)) = (1,0)$, $SE_2(P) = 0.45$, $x^*\left(P^{\{1\}}\right) = 1$, and $x^*\left(P^{\{2\}}\right) = 0.9$. Thus, x^* does not satisfy *se* and *ec*. ■

Proof of Theorem 2. (1) By Proposition 2, x^* satisfies both properties.

We now prove the uniqueness. Let f be a rule satisfying *cs* and *nas*.

Given a problem P, we know that there exists $s \in \mathbb{N}$ such that $x_j^{*1}(P) = 1$ for all $j < s$, $0 < x_s^{*1}(P) \leq 1$, and $x_j^{*1}(P) = 0$ for all $j > s$ and

$$\frac{p_1}{w_1} > ... > \frac{p_s}{w_s} > \frac{p_{s+1}}{w_{s+1}}...$$

Let i be in N. We take $h_i \in \mathbb{N}$ such that

$$\frac{(1-\frac{1}{h_i})p_1^i + \sum\limits_{k\in N:\ k\neq i} p_1^k}{w_1} > ... > \frac{(1-\frac{1}{h_i})p_s^i + \sum\limits_{k\in N:\ k\neq i} p_s^2}{w_s} > \frac{(1-\frac{1}{h_i})p_{s+1}^i + \sum\limits_{k\in N:\ k\neq i} p_{s+1}^2}{w_{s+1}}... \tag{4}$$

Let $N \subset N'$ such that $|N'\backslash N| = h_i - 1$. We consider $P' = (N', M, W, w, p')$ such that $p'^i = \frac{p^i}{h_i}$, $p'^k = \frac{p^i}{h_i}$ for all $k \in N'\backslash N$ and $p'^k = p^k$ for all $k \in N/\{i\}$. By *nas*,

$$f_i^2(P) \geq f_i^2(P') + \sum_{k\in N'\backslash N} f_k^2(P'). \tag{5}$$

Furthermore, by (4),

$$v_{P'}^r(i) = \frac{u_1(x^*(P))}{h_i} \text{ and } v_{P'}^r(k) = \frac{u_1(x^*(P))}{h_i} \text{ for all } k \in N'\backslash N.$$

By *cs*,

$$f_i^2(P') \geq \frac{u_1(x^*(P))}{h_i} \text{ and } f_k^2(P') \geq \frac{u_1(x^*(P))}{h_i} \text{ for all } k \in N'\backslash N.$$

By (5),

$$f_i^2(P) \geq u_i(x^*(P)). \tag{6}$$

As (6) holds for all $i \in N$, by *ef*,

$$f_i^2(P) = u_i(x^*(P)) \text{ for all } i \in N.$$

(2) By Proposition 2, x^* satisfies the four properties.

We now prove the uniqueness. Let f be a rule satisfying the four properties. Since f satisfies ef, $f^1(P) = x^*$.

Let s be such that $f_j^1(P) = 1$ for all $j < s$, $0 < f_s^1(P) \leq 1$, and $f_j^1(P) = 0$ for all $j > s$.

We take $W_1 = w_1$. By cu,

$$\begin{aligned} f_j^1(P) &= f_j^1(P(w_1)) + f_j^1\left(P\left(W - w_1, f^1(P(w_1))\right)\right) \text{ for all } j \in M \text{ and} \\ f_i^2(P) &= f_i^2(P(w_1)) + f_i^2\left(P\left(W - w_1, f^1(P(w_1))\right)\right) \text{ for all } i \in N. \end{aligned}$$

By ef,

$$f_j^1(P(w_1)) = \begin{cases} 1 & \text{if } j = 1 \\ 0 & \text{otherwise} \end{cases}$$

By iig

$$f(P(w_1)) = f\left(P(w_1)^{\{1\}}\right).$$

For each $i \in N$, $MA_i\left(P(w_1)^{\{1\}}\right) = p_1^i$. By ma $f_i^2\left(P(w_1)^{\{1\}}\right) \leq p_1^i$ for each $i \in N$. By ef, $\sum\limits_{i \in N} f_i^2\left(P(w_1)^{\{1\}}\right) = \sum\limits_{i \in N} p_1^i$. Thus,

$$f_i^2(P(w_1)) = f_i^2\left(P(w_1)^{\{1\}}\right) = p_1^i \text{ for each } i \in N.$$

We now apply cu to problem $P(W - w_1, f^1(P(w_1)))$ by taking $W_1 = w_2$. Let us make an abuse of notation and denote by $P(w_2)$ the first problem given by cu and by $P(W - w_1 - w_2)$ the second one. Using arguments similar to those used for $P(w_1)$ we can deduce. that

$$\begin{aligned} f_j^1(P(w_2)) &= \begin{cases} 1 & \text{if } j = 2 \\ 0 & \text{otherwise} \end{cases} \\ f_i^2(P(w_2)) &= p_2^i \text{ for each } i \in N. \end{aligned}$$

If we continue to apply cu we obtain that

$$\begin{aligned} f_j^1(P) &= \sum_{j=1}^{s-1} f_j^1(P(w_j)) + f_j^1\left(P\left(W - \sum_{j=1}^{s-1} w_j\right)\right) \text{ for all } j \in M \text{ and} \\ f_i^2(P) &= \sum_{j=1}^{s-1} f_i^2(P(w_j)) + f_i^2\left(P\left(W - \sum_{j=1}^{s-1} w_j\right)\right) \text{ for all } i \in N. \end{aligned}$$

where for each $j = 1, ..., s-1$,

$$
\begin{aligned}
f^1_{j'}\left(P\left(w_j\right)\right) &= \begin{cases} 1 & \text{if } j' = j \\ 0 & \text{otherwise} \end{cases} \\
f^2_i\left(P\left(w_j\right)\right) &= p^i_j \text{ for each } i \in N.
\end{aligned}
$$

and

$$
\begin{aligned}
f^1_j\left(P\left(W - \sum_{j=1}^{s-1} w_j\right)\right) &= \begin{cases} x^1_s\left(P\right) & \text{if } j = s \\ 0 & \text{otherwise} \end{cases} \\
f^2_i\left(P\left(W - \sum_{j=1}^{s-1} w_j\right)\right) &= p^i_s f^1_s\left(P\right) \text{ for each } i \in N.
\end{aligned}
$$

Thus, $f^1\left(P\right) = x^*$ and for each $i \in N$,

$$
\begin{aligned}
f^2_i\left(P\right) &= \sum_{j=1}^{s-1} f^2_i\left(P\left(w_j\right)\right) + f^2_i\left(P\left(W - \sum_{j=1}^{s-1} w_j\right)\right) \\
&= \sum_{j=1}^{s-1} p^i_j + p^i_s f^1_s\left(P\right) = \sum_{j=1}^{s-1} p^i_j x^{*1}_j\left(P\right) + p^i_s x^{*1}_s\left(P\right) \\
&= u_i\left(x^*\right).
\end{aligned}
$$

(3) By Proposition 2, x^* satisfies the properties.

We now prove the uniqueness by induction on n, the number of agents. Let f be a rule satisfying ef, ma and nas.

When $n = 1$, by ef, $f^1\left(P\right) = x^*$ and $f^2_1\left(P\right) = u_1\left(x^*\right)$.

We assume that $N = \{1, 2\}$. Given a problem P, let s be as in the definition of the optimal solution x^* given by 3. Since $P \in \mathcal{P}^*$,

$$
\frac{p^1_1 + p^2_1}{w_1} > ... > \frac{p^1_s + p^2_s}{w_s} > \frac{p^1_{s+1} + p^2_{s+1}}{w_{s+1}} ...
$$

Now, let $d_1 \in \mathbb{N}$ such that

$$
\frac{p^1_1 + (1 - \frac{1}{d_1})p^2_1}{w_1} > ... > \frac{p^1_s + (1 - \frac{1}{d_1})p^2_s}{w_s} > \frac{p^1_{s+1} + (1 - \frac{1}{d_1})p^2_{s+1}}{w_{s+1}} ... \qquad (7)
$$

Let $N \subset N'$ be such that $|N' \backslash N| = d_1 - 1$. We consider $P' = (N', M, W, w, p')$ such that $p'^1 = p^1$, $p'^2 = \frac{p^2}{d_1}$ and $p'^k = \frac{p^2}{d_1}$ for all $k \in N' \backslash N$. By *nas*,

$$f_2^2(P') + \sum_{k \in N' \backslash N} f_k^2(P') \leq f_2^2(P). \tag{8}$$

By *ef*,

$$f_1^2(P) \leq f_1^2(P'). \tag{9}$$

Now, let $P'' = (N'', M, W, w, p'')$ such that $N'' = \{1, 2\}$ and $p''^1 = p'^1 + \sum_{k \in N' \backslash N} p'^k$ and $p''^2 = p'^2 = \frac{p^2}{d_i}$.

Notice that P' is obtained from P'' when agent 2 in P'' split in agents $\{2\} \cup (N' \backslash N)$. By *nas*,

$$f_1^2(P') + \sum_{k \in N' \backslash N} f_k^2(P') \leq f_1^2(P''). \tag{10}$$

By (7),

$$MA_1(P'') = u_1(x^*(P'')).$$

By *ma*,

$$\begin{aligned} f_1^2(P'') \leq MA_1(P'') &= u_1(x^*(P'')) \\ &= \sum_{j \in M} p_j''^1 x_j^*(P'') \\ &= \sum_{j \in M} p_j^1 x_j^*(P'') + \sum_{j \in M} \sum_{k \in N' \backslash N} p'^k x_j^*(P'') \\ &= \sum_{j \in M} p_j^1 x_j^*(P') + \sum_{j \in M} \sum_{k \in N' \backslash N} p'^k x_j^*(P') \\ &= u_1(x^*(P')) + \sum_{k \in N' \backslash N} u_k(x^*(P')). \end{aligned} \tag{11}$$

By (10) and (11),

$$f_1^2(P') + \sum_{k \in N' \backslash N} f_k^2(P') \leq u_1(x^*(P')) + \sum_{k \in N' \backslash N} u_k(x^*(P')). \tag{12}$$

By (12) and *ef*,

$$f_2^2(P') \geq u_2(x^*(P')). \tag{13}$$

Similarly, if we take $\bar{k} \in N' \backslash N$ and consider $P''' = (N''', M, W, w, p''')$ such that $N'' = \{1, \bar{k}\}$ and $p'''^1 = p'^1 + p'^2 + \sum_{k \in N' \backslash \left(N \cup \{\bar{k}\}\right)} p'^k$ and $p'''^{\bar{k}} = p'^2 = \frac{p^2}{d_i}$, it can be proved that

$$f_{\bar{k}}^2 (P') \geq u_{\bar{k}} (x^*(P')) . \tag{14}$$

Then,

$$f_k^2 (P') \geq u_k (x^*(P')) \text{ for all } k \in N' \backslash N. \tag{15}$$

By (12) and (15),

$$f_1^2 (P') \leq u_1 (x^*(P')) . \tag{16}$$

By (9) and since $x^* (P) = x^* (P')$ and $p^1 = p'^1$,

$$f_1^2 (P) \leq u_1 (x^*(P)) .$$

Similarly it can be proved that

$$f_2^2 (P) \leq u_2 (x^*(P)) .$$

By ef,

$$f_i^2 (P) = u_i (x^*(P)) \text{ for all } i \in N.$$

We now consider the case $n \geq 3$. Assume that the result is true when we have less than n agents and we prove it for n.

We first prove that for any $P \in \mathcal{P}^*$ and any pair of agents $i, k \in N$ $(i \neq k)$

$$f_i^2 (P) + f_k^2 (P) \leq u_i (x^*(P)) + u_k (x^*(P)) . \tag{17}$$

We define $P^+ = (N^+, M, W, w, p^+)$ such that $N^+ = N \backslash \{k\}$ and $p^{+i} = p^i + p^k$ and $p^{+t} = p^t$ for all $t \in N^+ \backslash \{i\}$. By induction hypothesis

$$f_t^2 \left(P^+\right) = u_t \left(x^*(P^+)\right) \text{ for all } t \in N^+. \tag{18}$$

By nas,

$$f_i^2 (P) + f_k^2 (P) \leq f_i^2 \left(P^+\right) . \tag{19}$$

By (18) and (19)

$$f_i^2 (P) + f_k^2 (P) \leq u_i \left(x^*(P^+)\right) = u_i (x^*(P)) + u_k (x^*(P)) . \tag{20}$$

Fix $i \in N$, by 17

$$\sum_{k\in N\setminus\{i\}} \left[f_i^2(P) + f_k^2(P)\right] \leq \sum_{k\in N\setminus\{i\}} \left[u_i(x^*(P)) + u_k(x^*(P))\right] \Leftrightarrow$$
$$(n-1)f_i^2(P) + \sum_{k\in N\setminus\{i\}} f_k^2(P) \leq (n-1)u_i(x^*(P)) + \sum_{k\in N\setminus\{i\}} u_k(x^*(P)) \Leftrightarrow$$
$$(n-2)f_i^2(P) + \sum_{k\in N} f_k^2(P) \leq (n-2)u_i(x^*(P)) + \sum_{k\in N} u_k(x^*(P)). \tag{21}$$

By ef and since $n \geq 3$,

$$f_i^2(P) \leq u_i(x^*(P)). \tag{22}$$

Since (22) holds for all $i \in N$ and ef,

$$f_i^2(P) = u_i(x^*(P)).$$

We now prove that the properties used in the previous characterization are independent.

(1) Let $\bar{P}$ be the problem in Example (3). Let f^δ be such that $f^{\delta 1}(P) = x^*$ for each problem P. Besides, $f^{\delta 2}(P) = x^{*2}(P)$ if $P \neq \bar{P}$ and $f^2\left(\bar{P}\right) = (0.7, 2, 2)$. This rule satisfies cs, but fails nas.

Let f^γ be such that $f^{\gamma 1}(P) = x^*$ for each problem P. Besides, the total utility given by each good j is divided among the agents proportionally to the utility that each agent gives to the goods in x^*.Namely, given $i \in N$ and $j \in M$ we define:

$$y_j^i = \frac{\sum_{x_k^*>0} p_k^i}{\sum_{i\in N}\sum_{x_k^*>0} p_k^i} p_j x_j^*$$
$$f_i^{\gamma 2}(P) = \sum_{j\in M} y_j^i.$$

This rules satisfies nas but fails cs.

(2) Let f^0 be the rule that selects no good and allocates 0 to each agent. This rule satisfies ma, iig and cu but fails ef.

Let f^α be such that $f^{\alpha 1}(P) = x^*$ for each problem P. Besides, the total utility given by each good is divided equally among the agents given positive utility to such good. Namely, given $i \in N$ and $j \in M$ we define:

$$\begin{aligned}
N_j &= \left\{ i \in N : p_j^i > 0 \right\}. \\
y_j^i &= \begin{cases} \frac{1}{|N_j|} p_j x_j^* & \text{if } i \in N_j \\ 0 & \text{otherwise} \end{cases} \\
f_i^{\alpha 2}(P) &= \sum_{j \in M} y_j^i.
\end{aligned}$$

This rule satisfies ef, iig and cu but fails ma.

Let f^{β} be such that $f^{\beta 1}(P) = x^*$ for each problem P. Besides, the total utility is divided as equal as possible among the agents in such a way that no agent gets more than his maximum aspiration. Namely, given a problem P and $i \in N$,

$$f_i^{\beta 2}(P) = \min \{ MA_i(P), \alpha \} \text{ where } \sum_{i \in N} f_i^2(P) = \sum_{i \in N} u_i(x^*)^{2}.$$

This rule satisfies ef, ma, and cu but fails iig.

Let f^{π} be such that $f^{\pi 1}(P) = x^*$ for each problem P. Given $i \in N$ and $j \in M$ we define:

$$\begin{aligned}
M^{\pi} &= \left\{ j \in M : x_j^* > 0 \right\}, \\
FS^{\pi}(P) &= \left\{ x : \sum_{j \in M} w_j x_j \leq W \text{ and } x_j = 0 \text{ if } j \notin M^{\pi} \right\} \\
y^i &= \max_{x \in FS^{\pi}(P)} u_i(x)
\end{aligned}$$

Now, suppose that $N = \{i_1, ... i_n\}$ such that $y^{i_1} \geq y^{i_2} ... \geq y^{i_n}$. Notice that

[2] Notice that $f^{\beta 2}$ is defined as the constrained equal awards rule where the estate is the total utility of x^* and the claims are the maximum aspirations.

$u_i(x^*) \leq y_i \leq MA_i(P)$ for all $i \in N$. We define

$$\begin{aligned}
f_{i_1}^{\pi 2}(P) &= \min\{y_j^{i_1}, \sum_{i\in N} u_i(x^*)\}. \\
f_{i_2}^{\pi 2}(P) &= \min\{y_j^{i_2}, \sum_{i\in N} u_i(x^*) - f_{i_1}^{\pi 2}(P)\}. \\
&\vdots \\
f_{i_h}^{\pi 2}(P) &= \min\{y_j^{i_h}, \sum_{i\in N} u_i(x^*) - \sum_{r=1}^{h-1} f_{i_r}^{\pi 2}(P)\}. \\
&\vdots \\
f_{i_n}^{\pi 2}(P) &= \sum_{i\in N} u_i(x^*) - \sum_{r=1}^{n-1} f_{i_r}^{\pi 2}(P).
\end{aligned}$$

This rule satisfies ef, ma, and iig but fails cu.
(3) f^0 satisfies ma and nas but fails ef.
f^β satisfies ef and ma but fails nas.
f^γ satisfies ef and nas but fails ma. ■

Proof of Proposition 3. (1) It is obvious that Sh^o satisfies ef.

sym. Assume that agents i and j are symmetric in P. Thus, they are symmetric in the optimistic game v^o. Since the Shapley value satisfies symmetry, both agents receive the same. Thus Sh^o satisfies *sym*.

mon. Let P, P' and i as in the definition of *mon*. Since the Shapley value is an average of marginal contributions, it is enough to prove that for each $S \subset N\backslash\{i\}$, we have that

$$v_P^o(S\cup\{i\}) - v_P^o(S) \leq v_{P'}^o(S\cup\{i\}) - v_{P'}^o(S).$$

Since $v_{P'}^o(S) = v_P^o(S)$ it is enough to prove that $v_P^o(S\cup\{i\}) \leq v_{P'}^o(S\cup\{i\})$. Notice that $FS(P) = FS(P')$. Let $y \in FS(P)$ be such that $v_P^o(S\cup\{i\}) = \sum_{k\in S\cup\{i\}} \sum_{j\in M} p_j^k y_j$. Now,

$$\begin{aligned}
v_P^o(S\cup\{i\}) &= \sum_{k\in S\cup\{i\}} \sum_{j\in M} p_j^k y_j \leq \sum_{k\in S\cup\{i\}} \sum_{j\in M} p_j^{\prime k} y_j \\
&\leq \max_{x\in FS(P')} \sum_{k\in S\cup\{i\}} \sum_{j\in M} p_j^{\prime k} x_j = v_{P'}^o(S\cup\{i\}).
\end{aligned}$$

dum. Assume that agent i is a dummy in P. Thus, agent i is a dummy in the optimistic game v^o. Since the Shapley value satisfies dummy, agent i receives nothing. Thus Sh^o satisfies *dum*.

ma. Since the Shapley value is an average of marginal contributions, it is enough to prove that for each problem P, each $i \in N$, and each $S \subset N \backslash \{i\}$ we have that $v_P^o(S \cup \{i\}) - v_P^o(S) \leq MA_i(P)$.

Let $y, y' \in FS(P)$ be such that $v_P^o(S \cup \{i\}) = \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j^k y_j$ and $v_P^o(S) = \sum_{k \in S} \sum_{j \in M} p_j^k y_j'$. Now,

$$\begin{aligned} v_P^o(S \cup \{i\}) - v_P^o(S) &= \sum_{k \in S \cup \{i\}} \sum_{j \in M} p_j^k y_j - \sum_{k \in S} \sum_{j \in M} p_j^k y_j' \\ &= \sum_{j \in M} p_j^i y_j + \sum_{k \in S} \sum_{j \in M} p_j^k y_j - \sum_{k \in S} \sum_{j \in M} p_j^k y_j' \end{aligned}$$

By definition of y', $\sum_{k \in S} \sum_{j \in M} p_j^k y_j - \sum_{k \in S} \sum_{j \in M} p_j^k y_j' \leq 0$. Then,

$$v_P^o(S \cup \{i\}) - v_P^o(S) \leq \sum_{j \in M} p_j^i y_j \leq \max_{x \in FS(P)} \sum_{j \in M} p_j^i x_j = MA_i(P).$$

se. Let P be a problem and $i \in N$. Since $v^o(i) = SE_i(P)\, n$ and $v^o(S \cup i) \geq v^o(S)$ we have that $Sh_i^{o2}(P) \geq SE_i(P)$.

ec. Let P be a problem and $i, k \in N$. Let (N, v_P^o) be the corresponding optimistic game. Myerson (1980) proved that the Shapley value satisfies equal contributions in TU games. Then,

$$Sh_i(N, v_P^o) - Sh_i(N \backslash \{k\}, v_P^o) = Sh_k(N, v_P^o) - Sh_k(N \backslash \{i\}, v_P^o).$$

Since $Sh_i^{o2}(P) = Sh_i(N, v_P^o)$ and $Sh_k^{o2}(P) = Sh_k(N, v_P^o)$, it is enough to prove that $Sh_i^{o2}\left(P^{N \backslash \{k\}}\right) = Sh_i(N \backslash \{k\}, v_P^o)$ and $Sh_k^{o2}\left(P^{N \backslash \{i\}}\right) = Sh_k(N \backslash \{i\}, v_P^o)$. We prove that $Sh_i^{o2}\left(P^{N \backslash \{k\}}\right) = Sh_i(N \backslash \{k\}, v_P^o)$ (the other case is similar and we omit it). Since $Sh_i^{o2}\left(P^{N \backslash \{k\}}\right) = Sh_i\left(N \backslash \{k\}, v_{P^{N \backslash \{k\}}}^o\right)$, it is enough to prove that for each $T \subset N \backslash \{k\}$, $v_P^o(T) = v_{P^{N \backslash \{k\}}}^o(T)$. Notice that,

$$FS(P) = \left\{ x : \sum_{j \in M} w_j x_j \leq W \text{ and } x_j \in [0,1] \forall\ j \in M \right\} = FS\left(P^{N \backslash \{k\}}\right).$$

Then,

$$v_P^o(T) = \max_{x \in FS(P)} \sum_{i \in T} u_i(x) = \max_{x \in FS\left(P^{N \setminus \{k\}}\right)} \sum_{i \in T} u_i(x) = v^o_{P^{N\setminus\{k\}}}(S).$$

(2) It is obvious that Sh^o does not satisfy iig.

Consider Example 1. Since $v^o(1) = p_a^1$ and $v^o(S \cup 1) = v^o(S)$ when $\varnothing \neq S \subset N \setminus \{1\}$, we have that $Sh_1^{o2}(P) = Sh_1(v^o) = \frac{1}{3}p_a^1$. We take $W_1 = 2$. Since

$$\begin{aligned} v^o_{P(W_1)}(1) &= v^o_{P(W-W_1, Sh^{o1}(P(W_1)))}(1) = p_a^1, \\ v^o_{P(W_1)}(S \cup \{1\}) &= v^o_{P(W_1)}(S) \text{ and} \\ v^o_{P(W-W_1, Sh^{o1}(P(W_1)))}(S \cup \{1\}) &= v^o_{P(W-W_1, Sh^{o1}(P(W_1)))}(S) \text{ when } \varnothing \neq S \subset N \setminus \{1\}, \end{aligned}$$

we have that

$$Sh_1\left(v^o_{P(W-W_1, Sh^{o1}(P(W_1)))}\right) = Sh_1\left(v^o_{P(W_1)}\right) = \frac{1}{3}p_a^1.$$

Since,

$$Sh_1^{o2}(P(W_1)) + Sh_1^{o2}\left(P\left(W - W_1, Sh^{o1}(P(W_1))\right)\right) = \frac{2}{3}p_a^1,$$

we deduce that Sh^o does not satisfies cu.

Since Sh^o satisfies se and Proposition 1, we have that Sh^o does not satisfy cs.

nas. It follows from Theorem 2 (3) and the fact that the Sh^o satisfies ef and ma.

nam. Let P be such that $N = \{1, 2, 3\}$, $M = \{a, b, c\}$, $W = 1$ and $w_j = 1$ for all $j \in M$. Besides the vector p satisfies the following conditions: $p_a^1 = \frac{1}{2}$, $p_b^1 = 0$, $p_c^1 = 1$, $p_a^2 = 1$, $p_b^2 = 1$, $p_c^2 = 0$, $p_a^3 = \frac{3}{4}$, $p_b^3 = 1$ and $p_c^3 = 1$. Thus,

T	$v_P^o(T)$
$\{1\}$	1
$\{2\}$	1
$\{3\}$	1
$\{1,2\}$	$\frac{3}{2}$
$\{1,3\}$	2
$\{2,3\}$	2
N	$\frac{9}{4}$

Then, $Sh^{o2}(P) = Sh(v_P^o) = \left(\frac{2}{3}, \frac{2}{3}, \frac{11}{12}\right)$. Therefore, $Sh_1^{o2}(P) + Sh_2^{o2}(P) = \frac{4}{3}$.

Assume that agents 1 and 2 merge in agent 1. Now $N^+ = \{1, 3\}$, $p^{+1} = p^1 + p^2$, and $p^{+3} = p^3$. Then,

T	$v_{P+}^o(T)$
$\{1\}$	$\frac{3}{2}$
$\{3\}$	1
N^+	$\frac{9}{4}$

Then $Sh^{o2}(P^+) = Sh\left(v_{P+}^o\right) = \left(\frac{11}{8}, \frac{7}{8}\right)$.Then,

$$Sh_1^{o2}(P) + Sh_2^{o2}(P) = \frac{4}{3} < \frac{11}{8} = Sh_1^{o2}\left(P^+\right),$$

which implies Sh^o does not satisfy nam. ■

Proof of Theorem 3. By Proposition 3 we know that Sh^o satisfies ef and ec.

We now prove the uniqueness. This proof is quite standard in the literature. Let f be a rule satisfying ef and ec. We prove it by induction on n.

When $n = 1$, by ef, $f^1(P) = x^*$ and $f_1^2(P) = u_1(x^*)$. Assume that the result is true when we have less than n agents and we prove it for n. By ec, for all $i \in N \backslash \{1\}$,

$$\begin{aligned}
f_i^2(P) - f_i^2\left(P^{N\backslash\{1\}}\right) &= f_1^2(P) - f_1^2\left(P^{N\backslash\{i\}}\right) \Rightarrow \\
f_i^2(P) - f_1^2(P) &= f_i^2\left(P^{N\backslash\{1\}}\right) - f_1^2\left(P^{N\backslash\{i\}}\right) \Rightarrow \\
\sum_{i\in N\backslash\{1\}} f_i^2(P) - (n-1) f_1^2(P) &= \sum_{i\in N\backslash\{1\}} \left(f_i^2\left(P^{N\backslash\{1\}}\right) - f_1^2\left(P^{N\backslash\{i\}}\right)\right) \Rightarrow \\
\sum_{i\in N} f_i^2(P) - n f_1^2(P) &= \sum_{i\in N\backslash\{1\}} \left(f_i^2\left(P^{N\backslash\{1\}}\right) - f_1^2\left(P^{N\backslash\{i\}}\right)\right).
\end{aligned}$$

Since f satisfies ef, $\sum_{i\in N} f_i^2(P) = \sum_{i\in N} u_i(x^*)$.By induction hypothesis, $\sum_{i\in N\backslash\{1\}} \left(f_i^2\left(P^{N\backslash\{1\}}\right) - f_1^2\left(P^{N\backslash\{i\}}\right)\right)$ is known. Then,

$$f_1^2(P) = \frac{\sum_{i\in N} u_i(x^*) - \sum_{i\in N\backslash\{1\}} \left(f_i^2\left(P^{N\backslash\{1\}}\right) - f_1^2\left(P^{N\backslash\{i\}}\right)\right)}{n}.$$

Thus, $f_1^2(P)$ is uniquely determined. Let $i \in N \setminus \{1\}$. By ec,

$$f_i^2(P) = f_i^2\left(P^{N\setminus\{1\}}\right) + f_1^2(P) - f_1^2\left(P^{N\setminus\{i\}}\right),$$

which means that $f_i^2(P)$ is uniquely determined.

We now prove that the properties are independent.

f^0, defined as in Remark in the proof of Theorem 2, satisfies ec but fails ef.

f^β, defined as in Theorem 2, satisfies ef but fails ec. ■

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