ECOBAS Working Papers

2016 - 08

Title:

INDIVIDUALLY RATIONAL RULES FOR THE DIVISION PROBLEM WHEN THE NUMBER OF UNITS TO BE ALLOTTED IS ENDOGENOUS

Authors:

Gustavo Bergantiños
Universidade de Vigo

Jordi Massó
Universitat Autònoma de Barcelona

Alejando Neme
Universidad Nacional de San Luis and CONICET

ECOBAS
Economics and Business Administration for Society
www.ecobas.es

Individually Rational Rules for the Division Problem when the Number of Units to be Allotted is Endogenous*

Gustavo Bergantiños†, Jordi Massó‡ and Alejandro Neme§

June 27, 2016

Abstract: We study individually rational rules to be used to allot, among a group of agents, a perfectly divisible good that is freely available only in whole units. A rule is individually rational (at a preference profile) if each agent finds that her allotment is at least as good as any whole unit of the good. We study and characterize two individually rational and efficient rules, whenever agents' preferences are symmetric single-peaked on the set of possible allotments. The two rules are in addition envy-free, but they differ on wether envy-freeness is considered on losses or on awards. Our main result states that (i) the constrained equal losses rule is the unique individually rational and efficient rule that satisfies justified envy-freeness on losses and (ii) the constrained equal awards rule is the unique individually rational and efficient rule that satisfies envy-freeness on awards.

*The work of G. Bergantiños is partially supported by grants ECO2014-52616-R from the Spanish Ministry of Science and Competitiveness, GRC 2015/014 from "Xunta de Galicia", and 19320/PI/14 from "Fundación Séneca de la Región de Murcia". J. Massó acknowledges financial support from the Spanish Ministry of Economy and Competitiveness, through the Severo Ochoa Programme for Centres of Excellence in R&D (SEV-2015-0563) and grant ECO2014-53051, and from the Generalitat de Catalunya, through grant SGR2014-515. The paper was partly written while J. Massó was visiting the Department of Economics at Stanford University and acknowledges its hospitality as well as financial support received from the Spanish Ministry of Education, Culture and Sport, through project PR2015-00408. The work of A. Neme is partially supported by the Universidad Nacional de San Luis, through grant 319502, and by the Consejo Nacional de Investigaciones Científicas y Técnicas (CONICET), through grant PIP 112-200801-00655.

†Facultade de Económicas, Universidade de Vigo. 36310, Vigo (Pontevedra), Spain. E-mail: gbergant@uvigo.es

‡Universitat Autònoma de Barcelona and Barcelona Graduate School of Economics. Departament d'Economia i d'Història Econòmica, Campus UAB, Edifici B. 08193, Bellaterra (Barcelona), Spain. E-mail: jordi.masso@uab.es

§Instituto de Matemática Aplicada de San Luis. Universidad Nacional de San Luis and CONICET. Ejército de los Andes 950. 5700, San Luis, Argentina. E-mail: aneme@unsl.edu.ar

Journal of Economic Literature Classification Number: D71.
Keywords: Division problem; Single-peaked preferences; Individual rationality; Efficiency; Strategy-proofness; Envy-freeness.

1 Introduction

Consider the allotment problem faced by a group of agents who may share an homogeneous and perfectly divisible good, available only in integer units. For instance, a bound or a lottery ticket with (a potentially large) face value. Agents' risk attitudes and wealth induce single-peaked preferences on their potential allotments, the set of non-negative real numbers. A solution of the problem is a rule that selects, for each profile of agents' preferences (a profile for short), an integer number of units of the good to be allotted and a vector of allotments (one for each agent) whose sum is equal to this integer. But, for most profiles, the sum of agents' best allotments will be either larger or smaller than any integer number and hence, an endogenous rationing problem emerges, positive or negative depending on whether the chosen integer is smaller or larger to the sum of agents' best allotments. Sprumont (1991) studied the problem when the amount of the good to be allotted is fixed. He characterized the uniform rule as the unique efficient, strategy-proof and anonymous rule, on the domain of single-peaked preferences. The present paper can be seeing as an extension of Sprumont (1991)'s paper to a setting where the amount to be allotted of a perfectly divisible good has to be an integer, which may depend on agents' preferences.

We are interested in situations where the good is freely available to agents, but only in whole units. Hence, an agent will not accept a proposal of an allotment that is strictly worse than any integer amount of the good. For an agent with a (continuous) single-peaked preference, the set of allotments that are at least as good as any integer amount of the good (the set of individually rational allotments) is a closed interval that contains the best allotment, that we call the peak, and at least one of the two extremes of the interval is an integer. If preferences are symmetric, the peak is in the middle of the interval.

Our main concern then is to identify rules that select, for each profile of agents' symmetric single-peaked preferences, a vector of individually rational allotments. We call such rules individually rational. But since the set of individually rational rules is extremely large, and some of them are arbitrary and non-interesting, we would like to focus further on rules that are also efficient, strategy-proof, and satisfy some minimal fairness requirement. A rule is efficient if it selects, at each profile, a Pareto optimal vector of allotments: no other choices of (i) integer unit of the good to be allotted or (ii) vector of allotments, or (iii) both, can make all agents better off, and at least one of them strictly better off. We characterize the class of all efficient rules by means of two properties. First, the allotted amount of the good is the closest integer to the sum of agents' peaks. Second, all agents are rationed in the same direction: all receive more than their peaks, if the integer to be allotted is larger than the sum of the peaks, or all receive less, otherwise. A rule is strategy-proof if it induces, at each profile, truth-telling as a weakly dominant strategy in its associated direct revelation game. Our fairness requirement will be related to two alternative and well-known notions of envy-freeness, that we will adapt to our setting (justified envy-freeness from losses and envy-freeness from awards).[1]

We first show that there is no rule that is simultaneously efficient and strategy-proof.[2] We then proceed by studying separately two subclasses of rules; those that are individually rational and efficient and those that are individually rational and strategy-proof. For the first subclass, we identify the constrained equal losses rule and the constrained equal awards rule as the unique rules that, in addition of being individually rational and efficient, satisfy also either justified envy-freeness on losses or envy-freeness on awards, respectively. These rules divide the efficient integer amount of the good in such a way that all agents experience either equal losses or equal gains, subject to the constraint that all allotments have to be individually rational. Specifically, the constrained equal losses rule, evaluated at a profile, selects first the efficient number of integer units. Then, to allot this integer amount it proceeds with the rationing from the vector of peaks, by either reducing or increasing the peak of each agent (depending on whether the sum of the peaks is larger or smaller than the integer amount to be allotted) until the total amount is allotted. However, it makes sure that the extremes of agents' individually rational intervals are not overcome by excluding any agent from the rationing process as soon as one of the extremes of the agent's individually rational interval is reached, and it continues with the rest. The constrained equal awards rule is defined similarly but instead it uses, as the starting vector of the rationing process, either the vector of lower bounds or the vector of upper bounds of the individually rational intervals, depending on whether the sum of the peaks is larger or smaller than the integer amount to be allotted,

[1] See Thomson (2010) for a survey on envy-freeness.

[2] This is in contrast with Sprumont (1991)'s setting, which admits an extremely large class of efficient and strategy-proof rules. See Barberà, Jackson and Neme (1997) for a characterization of the set of sequential allotment rules as the class of all efficient, strategy-proof and replacement monotonic rules. To our knowledge this is the largest subclass of efficient and strategy-proof rules, on the domain of single-peaked profiles, characterized so far.

but makes sure that no agent's peak is overcome by excluding her from the rationing process as soon as her peak is reached, and it continues with the rest.

For the subclass of individually rational and strategy-proof rules, we show in contrast that although there are many rules satisfying the two properties simultaneously, they are not very interesting; for instance, none of them is unanimous. A rule is unanimous if, whenever the sum of the peaks is an integer, the rule selects this integer and it allots it according to the agents' peaks. We show then that individually rationality and strategy-proofness are indeed incompatible with unanimity.

At the end of the paper we extend some of our general and possibility results to the case where agents' preference are not necessarily symmetric. We argue why relevant strategy-proof rules in the classical division problem (i.e., the uniform rule and all sequential dictator rules) are not satisfactory in our setting. In particular, we show first that the (extended) uniform rule is efficient on the domain of all single-peaked preference profiles but it is neither strategy-proof nor individually rational, even in the domain of symmetric single-peaked preference profiles.[3] Finally, we show that all sequential dictator rules are efficient on the domain of all symmetric single-peaked preference profiles but they are neither individually rational nor strategy-proof, even in this subdomain.[4]

Before finishing this Introduction we mention some of the most related papers to ours. As we have already said, Sprumont (1991) proposed the division problem of a fixed amount of a good among a group of agents with single-peaked preferences on their potential allotments and provided two characterizations of the uniform rule, using strategy-proofness, efficiency and either anonymity or envy-freeness. Then, a very large literature followed Sprumont (1991) by taking at least two different paths. The first contains papers providing alternative characterizations of the uniform rule. See for instance Ching (1994) Sönmez (1994) and Thomson (1994a, 1994b, 1995 and 1997), whose characterizations we briefly discuss in the last section of the paper. The second group of papers proposed alternative rules when the problem is modified by introducing addi-

[3] The extended uniform rule allots, at each profile, the efficient integer amount as the uniform rule would do it. It is not strategy-proof because an agent may have incentives to missreport his preferences to induce a different choice of the integer amount, and it is not individually rational because the vector of allotments selected by the uniform rule is not individually rational in general.

[4] A sequential dictator rule, given a pre-specified order on the set of agents, proceeds by letting agents choose sequentially their peaks, rationing only the last agent whose allotment is the remainder amount that ensures that the sum of the allotments is equal to the efficient integer amount. Sequential dictator rules are not strategy-proof because the agent at the end of the ordering may have incentives to missreport her preference to induce a different amount to allot. They are not individually rational because the agent at the end of the ordering is rationed independently of his individually rational interval.

tional features or considering alternative domains of agents' preferences, or both. For instance, Ching (1992) extended the characterization of Sprumont (using envy-freeness) to the domain of single-plateaud preference profiles and Bergantiños, Massó and Neme (2012a, 2012b and 2015), Manjunath (2012) and Kim, Bergantiños and Chun (2015) study alternative ways of introducing individual rationality in the division problem. But in contrast with the present paper they assume that the quantity of the good to be allotted is fixed. Amorós (2002) started the multi-dimensional analysis of the division problem when several commodities have to be allotted among the same group of agents, but again the quantities of the goods to be allotted are fixed.

The paper is organized as follows. The next section presents the problem, preliminary notation and basic definitions. Section 3 contains the definitions of the properties of the rules that we will be concerned with. Section 4 describes the rules and states a preliminary result. Section 5 contains the main results of the paper for symmetric single-peaked preferences. Section 6 contains two final remarks.

2 The problem

We study situations where each agent of a finite set $N = \{1, \ldots, n\}$ wants an amount of a perfectly divisible good that can only be obtained in integer units and arbitrary portions of each unit can be freely allotted. We assume that $n \geq 2$ and denote by $x_i \geq 0$ the total amount of the good allotted to agent $i \in N$. Since all units of the good are alike, the amount x_i may come from different units. We assume that there is no limit on the (integer) number of units that can be allotted. Hence, and once N is fixed, the set of *feasible (vector of) allotments* is

$$FA = \{x = (x_1, \ldots, x_n) \in \mathbb{R}^N_+ \mid \sum_{i \in N} x_i \in \mathbb{N}\},$$

where $\mathbb{R}_+ = [0, +\infty)$ is the set of non-negative real numbers and $\mathbb{N} = \{1, 2, \ldots\}$ is the set of integers.[5]

Each agent i has a preference relation $\succeq_i$ defined on the set of potential allotments, which is a complete and transitive binary relation on $\mathbb{R}_+$. That is, for all $x_i, y_i, z_i \in \mathbb{R}_+$, either $x_i \succeq_i y_i$ or $y_i \succeq_i x_i$, and $x_i \succeq_i y_i$ and $y_i \succeq_i z_i$ imply $x_i \succeq_i z_i$; note that reflexivity ($x_i \succeq_i x_i$ for all $x_i \in \mathbb{R}_+$) is implied by completeness. Given $\succeq_i$, let $\succ_i$ be the antisymmetric binary relation on $\mathbb{R}_+$ induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \mathbb{R}_+$, $x_i \succ_i y_i$ if and only if $y_i \succeq x_i$ does not hold) and let $\sim_i$ be the indifference relation on $\mathbb{R}_+$

[5]Since no confusion can arise with negative integers, we refer to the set of non-negative integers $\mathbb{N}$ as the set of integers.

induced by $\succeq_i$ (*i.e.*, for all $x_i, y_i \in \mathbb{R}_+$, $x_i \sim_i y_i$ if and only if $x_i \succeq_i y_i$ and $y_i \succeq_i x_i$). We assume that $\succeq_i$ is continuous (*i.e.*, for each $x_i \in \mathbb{R}_+$ the sets $\{y_i \in \mathbb{R}_+ \mid y_i \succeq_i x_i\}$ and $\{y_i \in \mathbb{R}_+ \mid x_i \succeq_i y_i\}$ are closed) and that $\succeq_i$ is *single-peaked* on $\mathbb{R}_+$; namely, there exists a unique $p_i \in \mathbb{R}_+$, the *peak* of $\succeq_i$, such that $p_i \succ_i x_i$ for all $x_i \in \mathbb{R}_+ \backslash \{p_i\}$ and $x_i \succ_i y_i$ holds for any pair of allotments $x_i, y_i \in \mathbb{R}_+$ such that either $y_i < x_i \leq p_i$ or $p_i \leq x_i < y_i$. We say that agent i's single-peaked preference $\succeq_i$ is *symmetric* on $\mathbb{R}_+$ if for all $z_i \in [0, p_i]$, $(p_i - z_i) \sim_i (p_i + z_i)$; that is, for all $x_i, y_i \in \mathbb{R}_+$, $x_i \succeq_i y_i$ if and only if $|p_i - x_i| \leq |p_i - y_i|$. Notice two things. First, the peak of a symmetric single-peaked preference conveys all information about the whole preference. Thus, we will often identify a symmetric single-peaked preference $\succeq_i$ with its peak p_i. Second, for each $x_i \in \mathbb{R}_+$ there exists an integer $k_{x_i} \in \mathbb{N}$ such that $k_{x_i} \leq x_i < k_{x_i} + 1$. Hence, the following notation is well-defined:

$$\begin{aligned} [x_i]_l &= k_{x_i} \\ [x_i]_u &= k_{x_i} + 1, \text{ and} \\ [x_i] &= \begin{cases} k_{x_i} & \text{if } x_i \leq k_{x_i} + 0.5 \\ k_{x_i} + 1 & \text{if } x_i > k_{x_i} + 0.5. \end{cases} \end{aligned}$$

In particular, $[x_i]$ is the integer closest to x_i (if it is unique) and $[x_i] = k_{x_i}$ if $x_i = \frac{2k_{x_i}+1}{2}$.

A (division) *problem* is a pair $(N, \succeq)$ where N is the set of agents and $\succeq = (\succeq_1, \ldots, \succeq)$ is a profile of single-peaked preferences on $\mathbb{R}_+$, one for each agent in N. Since the set N will remain fixed we often write $\succeq$ instead of $(N, \succeq)$ and refer to $\succeq$ as a problem and as a profile, interchangeably. To emphasize agent i's preference $\succeq_i$ in the profile $\succeq$ we often write it as $(\succeq_i, \succeq_{-i})$.

We denote by $\mathcal{P}$ the set of all problems and by $\mathcal{P}^S$ the set of all problems where agents' preferences are symmetric single-peaked.

Since preferences are idiosyncratic, they have to be elicited. A *rule* on $\mathcal{P}$ is a function f assigning to each problem $\succeq \in \mathcal{P}$ a feasible allotment $f(\succeq) = (f_1(\succeq), \ldots, f_n(\succeq)) \in FA$. We will also consider rules defined only on $\mathcal{P}^S$. Any rule on $\mathcal{P}$ can be restricted to operate only on $\mathcal{P}^S$.

To study rules on $\mathcal{P}^S$ selecting individually rational allotments, the following intervals will play a critical role. Fix a problem $\succeq \in \mathcal{P}^S$, with its vector of peaks $(p_1, \ldots, p_n)$. For each $i \in N$, define the associated interval

$$[l_i, u_i] = \begin{cases} [[p_i]_l, p_i + (p_i - [p_i]_l)] & \text{if } [p_i] = [p_i]_l \\ [p_i - ([p_i]_u - p_i), [p_i]_u] & \text{if } [p_i] = [p_i]_u. \end{cases}$$

Allotments outside the interval $[l_i, u_i]$ are strictly worse to some integer allotment, and they will not be acceptable to i, if agents have free access to any integer amount of the good. Since each interval $[l_i, u_i]$ depends only on p_i, we call it the *individually rational interval of* p_i (Proposition 2 will show the exact relationship between individually rational rules on $\mathcal{P}^S$ and these individually rational intervals). Given $p_i \in \mathbb{R}_+$, $[l_i, u_i]$ can be seen as the unique interval with the properties that p_i is equidistant to the two extremes (i.e. $p_i = \frac{l_i+u_i}{2}$), at least one of the two extremes is an integer, and its length is at most one. For instance, the individually rational interval of $p_i = 1.8$ is $[1.6, 2]$ and of $p_i = 2.3$ is $[2, 2.6]$.

3 Properties of rules

We now describe possible properties that a rule f on $\mathcal{P}$ (or on $\mathcal{P}^S$) may satisfy. Again, the properties defined on $\mathcal{P}$ can be straightforwardly extended to $\mathcal{P}^S$ by restricting their definitions to the set of problems in $\mathcal{P}^S$.

We start with the property of individual rationality, the one that we found more basic for the class of problems we are interested in, which is the main focus of this paper. Since we are assuming that all integer units of the good are freely available, even for a single agent, a rule is individually rational if each agent considers his allotment at least as good as any integer number of units of the good.

Individual rationality. For all $\succeq \in \mathcal{P}$, $i \in N$ and $k \in \mathbb{N}$, $f_i(\succeq) \succeq_i k$.

The next two properties are also very appealing. Efficiency says that, for each problem, the allotments selected by the rule is Pareto undominated in the set of feasible allotments, while a rule is strategy-proof if agents can never obtain a strictly better allotment by misrepresenting their preferences.

Efficiency. For all $\succeq \in \mathcal{P}$, there does not exist $y \in FA$ such that $y_i \succeq_i f_i(\succeq)$ for all $i \in N$ and $y_j \succ f_j(\succeq)$ for at least one $j \in N$.

Strategy-proofness. For all $\succeq \in \mathcal{P}$, $i \in N$ and single-peaked preference $\succeq'_i$,

$$f_i(\succeq) \succeq_i f_i(\succeq'_i, \succeq_{-i}).$$

Agent i *manipulates* f *at* $\succeq$ *via* $\succeq'_i$ if $f_i(\succeq'_i, \succeq_{-i}) \succ_i f_i(\succeq)$.

We will also consider other desirable properties of rules. Participation says that agents will not have interest in obtaining integer units of the good in addition to their received allotments. To define it formally, we need some additional notation. For each $\succeq \in \mathcal{P}$, $i \in N$ and $k \in \mathbb{N}$ such that $k \leq p_i$, let $\succeq_i^{-k}$ be the single-peaked preference on $\mathbb{R}_+$ obtained from $\succeq_i$ by shifting it downwards in k units; namely, for each pair $x_i, y_i \in \mathbb{R}_+$, $x_i \succeq_i^{-k} y_i$ if and only if $k + x_i \succeq_i k + y_i$.

Participation. For all $\succeq \in \mathcal{P}$, $i \in N$ and $k \in \mathbb{N}$ such that $k \leq p_i$,

$$f_i(\succeq) = k + f_i(\succeq_i^{-k}, \succeq_{-i}).$$

Unanimity says that the rule selects the profile of peaks whenever it is a feasible vector of allotments. Equal treatment of equals says that agents with the same preferences receive equal allotments.

Unanimity. For all $\succeq \in \mathcal{P}$ such that $\sum_{j \in N} p_j \in \mathbb{N}$, $f_i(\succeq) = p_i$ for all $i \in N$.

Equal treatment of equals. For all $\succeq \in \mathcal{P}$ and $i, j \in N$ such that $\succeq_i = \succeq_j$, $f_i(\succeq) = f_j(\succeq)$.

Envy-freeness says that the rule selects a vector of allotments with the property that no agent would strictly prefer the allotment of another agent.

Envy-freeness. For all $\succeq \in \mathcal{P}$ and $i, j \in N$, $f_i(\succeq) \succeq_i f_j(\succeq)$.

The next three properties are alternative versions of envy-freeness, once adapted to our context when agents have symmetric single-peaked preferences and they have free access to any integer amount of the good. The emphasis is on either the losses or the awards that agents' allotments represent with respect to either their peaks or the extremes of their individually rational intervals, respectively. First, envy-freeness on losses says that each agent prefers his loss (with respect to his peak) to the loss of any other agent.

Envy-freeness on losses. For all $\succeq \in \mathcal{P}^S$ and $i, j \in N$, $f_i(\succeq) \succeq_i \max\{p_i + (f_j(\succeq) - p_j), 0\}$.[6]

Second, justified envy-freeness on losses qualifies the previous property by requiring that each agent i prefers his loss (*i.e.*, $f_i(\succeq) - p_i$) to the loss of any other agent j (*i.e.*, $f_j(\succeq) - p_j$), only if j's allotment is non-integer. Since agents can obtain freely any integer number of units of the good, it may be understood that it is not legitimate to express envy against another agent who is receiving an integer allotment because it is as if the rule would not allot to this other agent any amount.

Justified envy-freeness on losses. For all $\succeq \in \mathcal{P}^S$ and $i, j \in N$ such that $f_j(\succeq) \notin \mathbb{N}$, $f_i(\succeq) \succeq_i \max\{p_i + (f_j(\succeq) - p_j), 0\}$.

Envy-freeness on awards roughly says that each agent prefers her award, with respect to her individual rational allotment, to any amount between her award and the award of any other agent. To state it formally, let f be a rule on $\mathcal{P}^S$. Define, for each $\succeq \in \mathcal{P}^S$

[6]Note that $f_i(\succeq) = p_i + (f_i(\succeq) - p_i)$ always holds; hence, the condition in the definition is trivially satisfied whenever $i = j$.

and $i \in N$, the award of i (at $(\succeq, f)$) with respect to i's individual rational interval as

$$a_i(\succeq, f) = \begin{cases} f_i(\succeq) - l_i & \text{if } f_i(\succeq) \leq p_i \\ u_i - f_i(\succeq) & \text{if } f_i(\succeq) > p_i. \end{cases}$$

When no confusion arises we write a_i instead of $a_i(\succeq, f)$.

Envy-freeness on awards. For all $\succeq \in \mathcal{P}^S$ and $i, j \in N$,

$$x \in [\min\{a_i(\succeq, f), a_j(\succeq, f)\}, \max\{a_i(\succeq, f), a_j(\succeq, f)\}]$$

implies $f_i(\succeq) \succeq_i l_i + x$.[7]

Example 1 might help to better understand this property.

Example 1 Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2, 3\}$, $p = (0.1, 0.6, 0.6)$ and $f(\succeq) = (0, 0.5, 0.5)$. Then, $l_1 = 0$, $l_2 = 0.2$, $a_1(\succeq, f) = 0$, and $a_2(\succeq, f) = 0.3$ and

$$[\min\{a_1(\succeq, f), a_2(\succeq, f)\}, \max\{a_1(\succeq, f), a_2(\succeq, f)\}] = [0, 0.3].$$

By setting $x = 0.3$ we have that $f_1(\succeq) = 0 \succeq_1 0.3 = l_1 + x$. Nevertheless, by setting $x = 0.1$ we have that $f_1(\succeq) = 0 \prec_1 0.1 = l_1 + x$ and so, f would not satisfy envy-freeness on awards. Notice that in this case agent 1 can argue that agent 3 is receiving (when comparing with the individual rational point) more than him. □

Finally, group rationality is an extension of individual rationality to groups of agents. It says that each subset of agents receives a total allotment that is "at least as good" (in aggregate terms) as any other total allotment they could receive only by themselves.

Group rationality. For all $\succeq \in \mathcal{P}^S$, $S \subset N$ and $k \in \mathbb{N}$,

$$\left|\sum_{i \in S} p_i - \sum_{i \in S} f_i(\succeq)\right| \leq \left|\sum_{i \in S} p_i - k\right|.$$

Remark 1 The following statements hold.[8]

(R1.1) If f is efficient on $\mathcal{P}$, then f is unanimous.

(R1.2) If f is envy-free on losses on $\mathcal{P}^S$, then f satisfies justified envy-freeness on losses on $\mathcal{P}^S$.

(R1.3) If f is group rational on $\mathcal{P}^S$, then f is individually rational on $\mathcal{P}^S$.

(R1.4) Envy-freeness and envy-freeness on losses are unrelated.

[7] Since $\succeq_i$ is symmetric single-peaked, for all such x, $f_i(\succeq) \succeq_i l_i + x$ is equivalent to $f_i(\succeq) \succeq_i u_i - x$.

[8] The proof that (R1.1), (R1.2) and (R1.3) hold is inmediate. Example 2 indicates the main reasons why (R1.4) and (R1.5) hold as well.

(R1.5) Envy-freeness and envy-freeness on awards are unrelated.

Example 2 Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (1.4, 3.4)$. Any rule that selects the feasible vector of allotments $(1.6, 3.4)$ at $\succeq$ may satisfy envy-freeness but it would fail to satisfy envy-freeness on losses and envy-freeness on awards. Consider now the profile $\succeq' \in \mathcal{P}^S$ where $p' = (0.35, 0.45)$. Any rule that selects the feasible vector of allotments $(0.45, 0.55)$ at $\succeq'$ may satisfy envy-freeness on losses but it would fail to satisfy envy-freeness. Finally, consider the profile $\succeq'' \in \mathcal{P}^S$ where $p'' = (0.6, 0.8)$. Any rule that selects the feasible vector of allotments $(0.3, 0.7)$ at $\succeq''$ may satisfy envy-freeness on awards but it would fail to satisfy envy-freeness. □

4 Rules

In this section we adapt, to our setting with integer amounts, fair and well-known rules that have already been used to solve the division problem with a fixed amount. Since our main results will be relative to symmetric single-peaked preferences, we already restrict the rules we consider in the next two sections to operate on $\mathcal{P}^S$. This is important because the rules will allot the integer amount that is closest to the sum of the peaks, which is always the efficient amount only if single-peaked preferences are symmetric. Although we will be interested only on their constrained versions (to unsure that they are individually rational) we also present their unconstrained versions for further reference and because they may help the reader to understand the constrained ones. The first one is the equal losses rule f^{EL}. At any profile, f^{EL} selects the feasible vector of allotments by the following egalitarian procedure. Start from the vector of peaks and, if this is an unfeasible vector of allotments, increase (or decrease) all agents' allotments in the same amount until the integer $[\sum_{j \in N} p_j]$ is allotted, stopping the decrease (if this is the case) of any agent's allotment, as soon as the zero allotment is reached.

Equal losses $\left(f^{EL}\right)$. For all $\succeq \in \mathcal{P}^S$ and $i \in N$, set

$$f_i^{EL}(\succeq) = \begin{cases} p_i - \min\{\alpha, p_i\} & \text{if } \sum_{j \in N} p_j \geq [\sum_{j \in N} p_j] \\ p_i + \alpha & \text{if } \sum_{j \in N} p_j < [\sum_{j \in N} p_j], \end{cases}$$

where α is the unique real number for which $\sum_{j \in N} f_j^{EL}(\succeq) = [\sum_{j \in N} p_j]$ holds.[9]

Figure 1 represents f^{EL} at the profiles $\succeq$, $\succeq'$ and $\bar{\succeq}$, where $p_1 + p_2 = p_1' + p_2' > [p_1 + p_2] = [p_1' + p_2']$ and $\bar{p}_1 + \bar{p}_2 < [\bar{p}_1 + \bar{p}_2]$.

[9] Corollary 1 below (that follows from Proposition 1) will establish the existence of such unique real number α, as well as the existence of the real numbers $\widehat{\alpha}$, β, and $\widehat{\beta}$, used to define the other three rules below.

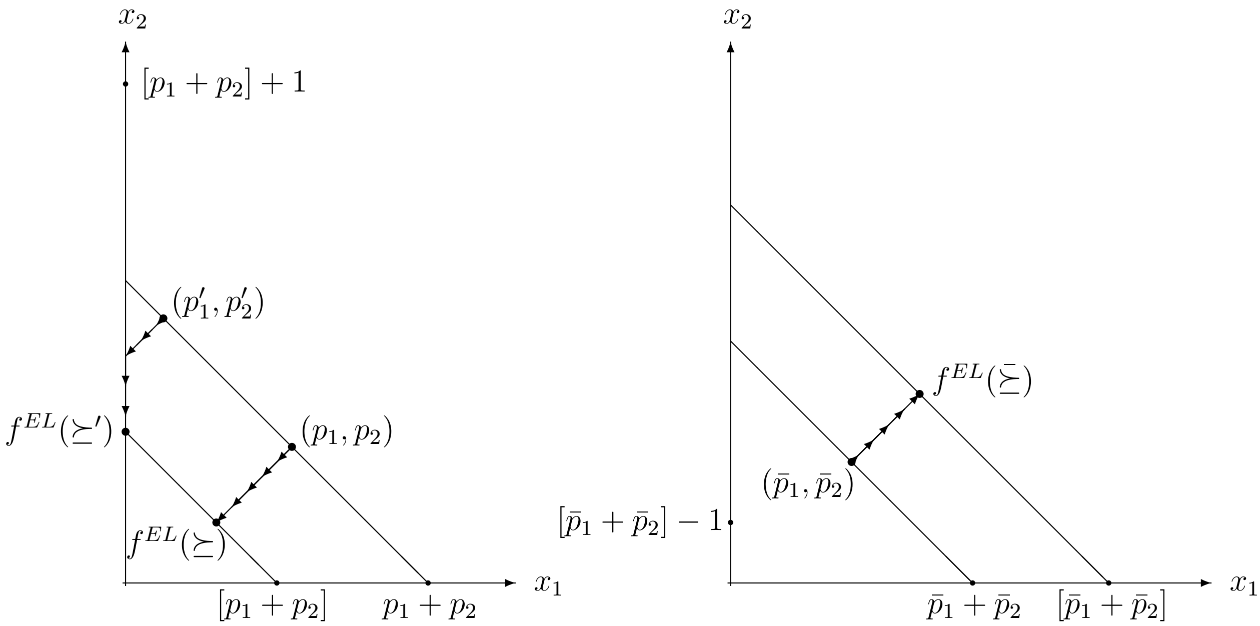

Figure 1

The constrained equal losses rule f^{CEL} proceeds by following the same egalitarian procedure but now the increase or decrease of the allotment of agent i, starting from p_i, stops as soon as i's allotment is equal to the relevant extreme of i's individually rational interval.

Constrained equal losses (f^{CEL}). For all $\succeq \in \mathcal{P}^S$ and $i \in N$, set

$$f_i^{CEL}(\succeq) = \begin{cases} p_i - \min\{\widehat{\alpha}, p_i - l_i\} & \text{if } \sum_{j \in N} p_j \geq [\sum_{j \in N} p_j] \\ p_i + \min\{\widehat{\alpha}, u_i - p_i\} & \text{if } \sum_{j \in N} p_j < [\sum_{j \in N} p_j], \end{cases}$$

where $\widehat{\alpha}$ is the unique real number for which $\sum_{j \in N} f_j^{CEL}(\succeq) = [\sum_{j \in N} p_j]$ holds.

Figure 2 represents f^{CEL} at the profiles $\succeq$ and $\bar{\succeq}$, where $p_1 + p_2 > [p_1 + p_2]$ and $\bar{p}_1 + \bar{p}_2 < [\bar{p}_1 + \bar{p}_2]$.

The equal awards rule f^{EA} follows the same egalitarian procedure used to define f^{EL}, but instead of starting from the vector of peaks, it starts from the vector of relevant extremes of the individual rational intervals and, it increases (or decreases) all agents' allotments in the same amount until the integer number of units $[\sum_{j \in N} p_j]$ is allotted, making sure that no agent receives a negative allotment.

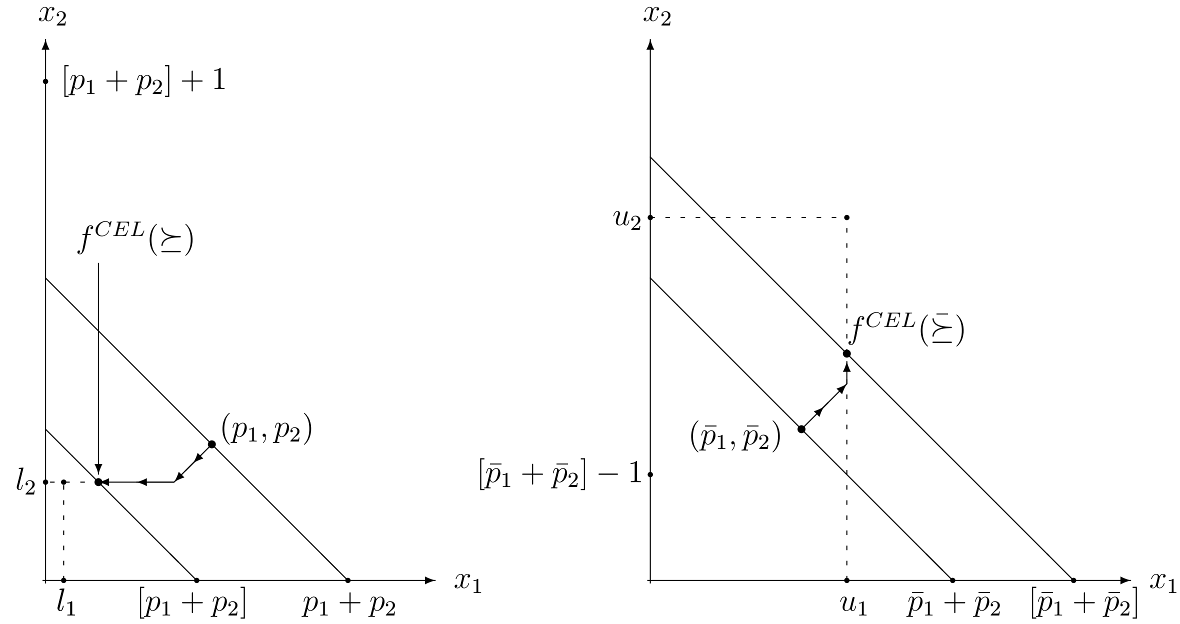

Figure 2

Equal awards $\left(f^{EA}\right)$. For all $\succeq \in \mathcal{P}^S$ and $i \in N$, set

$$f_i^{EA}(\succeq) = \begin{cases} l_i + \beta & \text{if } \sum_{j\in N} p_j \geq [\sum_{j\in N} p_j] \\ u_i - \min\{\beta, u_i\} & \text{if } \sum_{j\in N} p_j < [\sum_{j\in N} p_j], \end{cases}$$

where β is the unique real number for which $\sum_{j\in N} f_j^{EA}(\succeq) = [\sum_{j\in N} p_j]$ holds.

Figure 3 represents f^{EA} at the profiles $\succeq$, $\succeq'$ and $\bar{\succeq}$, where $p_1 + p_2 > [p_1 + p_2]$, $p_1' + p_2' < [p_1' + p_2']$, $\bar{p}_1 + \bar{p}_2 < [\bar{p}_1 + \bar{p}_2]$ and $[p_1' + p_2'] = [\bar{p}_1 + \bar{p}_2]$.

The constrained equal awards rule f^{CEA} proceeds by following the same egalitarian procedure used to describe f^{EA}, but now the increase or decrease of the allotment of each agent i, starting from the relevant extreme of i's individually rational interval, stops as soon as i's allotment is equal to p_i.

Constrained equal awards f^{CEA}. For all $\succeq \in \mathcal{P}^S$ and $i \in N$, set

$$f_i^{CEA}(\succeq) = \begin{cases} l_i + \min\{\widehat{\beta}, p_i - l_i\} & \text{if } \sum_{j\in N} p_j \geq [\sum_{j\in N} p_j] \\ u_i - \min\{\widehat{\beta}, u_i - p_i\} & \text{if } \sum_{j\in N} p_j < [\sum_{j\in N} p_j], \end{cases}$$

where $\widehat{\beta}$ is the unique real number for which $\sum_{j\in N} f_j^{CEA}(\succeq) = [\sum_{j\in N} p_j]$ holds.

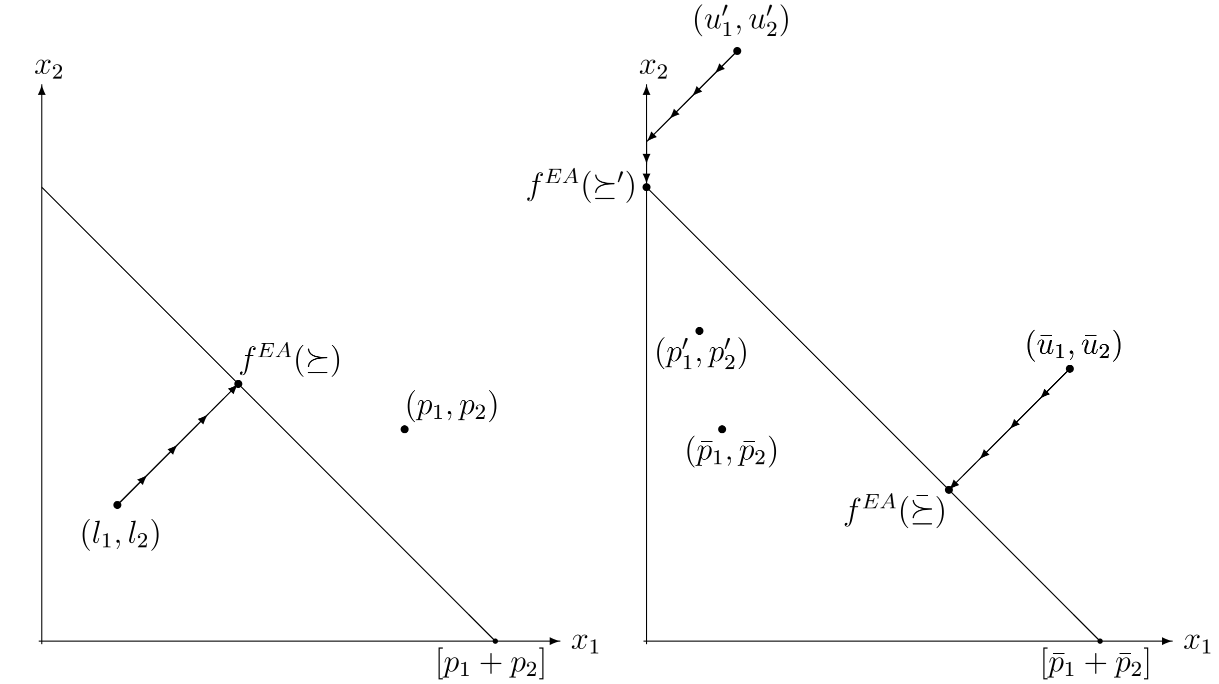

Figure 3

Figure 4 represents f^{CEA} at the profiles $\succeq$ and $\bar{\succeq}$, where $p_1 + p_2 > [p_1 + p_2]$ and $\bar{p}_1 + \bar{p}_2 < [\bar{p}_1 + \bar{p}_2]$.

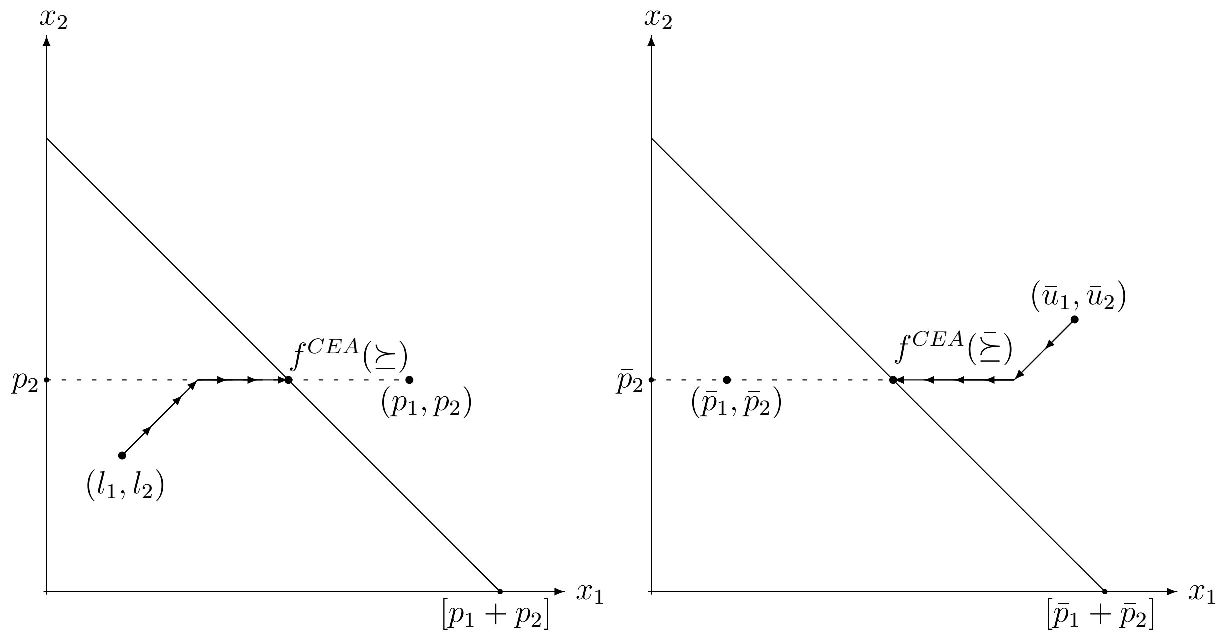

Figure 4

The existence of the unique numbers α, $\widehat{\alpha}$, β and $\widehat{\beta}$ in each of the above definitions is guaranteed by Proposition 1 below. The translation of its content at the figures representing the four rules is as follows. Conditions (P1.1) and (P1.2) guarantee that the vector of lower bounds $l = (l_1, \ldots, l_n)$ and the vector of upper bounds $u = (u_1, \ldots, u_n)$ lie below and above the hyperplane $\{y \in \mathbb{R}^N_+ \mid \sum_{j\in N} y_j = [\sum_{j\in N} p_j]\}$, respectively.[10]

Proposition 1 *For each $\succeq \in \mathcal{P}^S$, the relevant statement below holds.*

(P1.1) *If $\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j]$ then $\sum_{j\in N} l_j \leq [\sum_{j\in N} p_j]$.*

(P1.2) *If $\sum_{j\in N} p_j < [\sum_{j\in N} p_j]$ then $\sum_{j\in N} u_j \geq [\sum_{j\in N} p_j]$.*

Proof Let $\succeq \in \mathcal{P}^S$ be arbitrary.

(P1.1) To obtain a contradiction suppose

$$\textstyle\sum_{j\in N} l_j > [\sum_{j\in N} p_j]. \tag{1}$$

For all $i \in N$, $l_i \leq p_i$. Then, there exists at least one i such that $l_i < p_i$; otherwise, if for all $j \in N$, $l_j = p_j$ holds, then $p_j \in \mathbb{N}$ for all $j \in N$, and (1) could not hold. Hence, by the definition of $[\sum_{j\in N} p_j]$, and since $l_i \leq p_i$ for all $i \in N$ and $\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j]$, there exists $k \in \mathbb{N}$ such that

$$k = [\textstyle\sum_{j\in N} p_j] < \sum_{j\in N} l_j < \sum_{j\in N} p_j \leq k + 0.5. \tag{2}$$

For each $i \in N$ there exists $k_i \in \mathbb{N}$ such that $k_i \leq l_i \leq p_i < k_i + 1$. To see that note that given $p_i \in [l_i, u_i]$ either $l_i \in \mathbb{N}$ or $u_i \in \mathbb{N}$ (or both, if $p_i = \frac{l_i+u_i}{2}$) and so, if $l_i \in \mathbb{N}$ then $l_i = k_i$ and if $u_i \in \mathbb{N}$ then $u_i = k_i + 1$. Consider the symmetric single-peaked preference $\succeq_i^{k_i}$, defined as in the preference used in the definition of the participation property. Let l_i' and p_i' be the corresponding lower bound and peak associated with $\succeq_i^{k_i}$. Thus, $l_i' = l_i - k_i$ and $p_i' = p_i - k_i$. Note that $0 \leq p_i' < 1$. It is easy to see that the three equalities below hold.

$$\begin{aligned} [\textstyle\sum_{j\in N} p_j] &= \textstyle\sum_{j\in N} k_j + [\sum_{j\in N} p_j'], \\ \textstyle\sum_{j\in N} l_j &= \textstyle\sum_{j\in N} k_j + \sum_{j\in N} l_j', \text{ and} \\ \textstyle\sum_{j\in N} p_j &= \textstyle\sum_{j\in N} k_j + \sum_{j\in N} p_j'. \end{aligned}$$

Let $k' = k - \sum_{j\in N} k_j$. Then, by (2) and the equalities above,

$$k' = [\textstyle\sum_{j\in N} p_j'] < \sum_{j\in N} l_j' < \sum_{j\in N} p_j' \leq k' + 0.5.$$

Hence, without loss of generality, we can assume that $0 \leq p_i < 1$ for all $i \in N$. Let $S = \{i \in N \mid l_i = 0\}$. Observe that if $i \notin S$, $l_i < p_i$ and $u_i = 1$. Thus,

$$k = [\textstyle\sum_{j\in N} p_j] < \sum_{j\in N} l_j = \sum_{j\in N\setminus S} l_j < \sum_{j\in N\setminus S} p_j \leq \sum_{j\in N} p_j \leq k + 0.5.$$

[10]It is immediate to see that (i) if $\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j]$ then $\sum_{j\in N} u_j \geq [\sum_{j\in N} p_j]$ and (ii) if $\sum_{j\in N} p_j < [\sum_{j\in N} p_j]$ then $\sum_{j\in N} l_j < [\sum_{j\in N} p_j]$.

Hence, $[\sum_{j\in N\setminus S} p_j] = k$, and so we can also assume that $l_i > 0$ for all $i \in N$. Since $0 < p_i < 1$, $l_i < p_i$, and $u_i = 1$, we have that $\sum_{j\in N} p_j < n$. Then, $k \leq n-1$. Besides, $\sum_{j\in N} u_j = n$. By (2), $\sum_{j\in N}(p_j - l_j) < 0.5$. Since, by symmetry, $p_i - l_i = u_i - p_i$ for all $i \in N$, $\sum_{j\in N}(u_j - p_j) = \sum_{j\in N}(p_j - l_j) < 0.5$. But since, by (2), $\sum_{j\in N} p_j \leq k + 0.5$ holds,

$$n = \sum_{j\in N} u_j = \sum_{j\in N} p_j + \sum_{j\in N}(u_j - p_j) < k + 1 \leq n$$

holds as well, which is a contradiction.

(P1.2) The proof is analogous to the one used to prove part (P1.1), and hence we omit it. ■

Proposition 1 implies that the real numbers α, $\widehat{\alpha}$, β and $\widehat{\beta}$ used to define the four rules do exist, and hence the rules are well-defined. To see that, observe that f^{EL} and f^{CEL} start allotting the good from p in a continuous and egalitarian (or constrained egalitarian) way until the full amount $[\sum_{j\in N} p_j]$ is allotted. On the other hand, f^{EA} and f^{CEA} start allotting the good from the vector of relevant extremes of the individually rational intervals in a continuous and egalitarian (or constrained egalitarian) way until the full amount $[\sum_{j\in N} p_j]$ is allotted. Proposition 1 guarantees that the direction of the allotment process goes in the right direction to reach $[\sum_{j\in N} p_j]$, from either one of the two starting vectors. So, Corollary 1 holds.

Corollary 1 *The real numbers α, $\widehat{\alpha}$, β and $\widehat{\beta}$, used to define f^{EL}, f^{CEL}, f^{EA} and f^{CEA} respectively, do exist and they are unique.*

5 Results for symmetric single-peaked preferences

5.1 Individual rationality, efficiency and basic impossibilities

In the next proposition we present some results related with the properties of rules, whenever they operate on problems where agents' preferences are symmetric single-peaked. The first result characterizes individually rational rules by stating that a rule is individually rational if and only if, for all profiles, the rule selects a vector of allotments that lie on the individually rational intervals of their peaks. The second result characterizes efficient rules by means of two conditions. Firstly, at each profile the rule allots the integer amount that is closer to the sum of all peaks and secondly, all agents receive more (or less) than their peaks whenever the sum of all peaks is smaller (or larger) than the closest integer to the sum of peaks. We also show that some basic incompatibilities among properties of rules hold, even when agents' preferences are restricted to be symmetric single-peaked.

Proposition 2 *The following statements hold.*

(P2.1) *A rule f on $\mathcal{P}^S$ is individually rational if and only if, for all $\succeq \in \mathcal{P}^S$ and $i \in N$, $f_i(\succeq) \in [l_i, u_i]$.*

(P2.2) *A rule f on $\mathcal{P}^S$ is efficient if and only if, for all $\succeq \in \mathcal{P}^S$, two conditions hold:*

(E2.1) $\sum_{j\in N} f_j(\succeq) = \left[\sum_{j\in N} p_j\right]$.

(E2.2) *For all $i \in N$, $f_i(\succeq) \leq p_i$ when $\sum_{j\in N} p_j \geq \left[\sum_{j\in N} p_j\right]$ and $f_i(\succeq) \geq p_i$ when $\sum_{j\in N} p_j < \left[\sum_{j\in N} p_j\right]$.*

(P2.3) *There is no rule on $\mathcal{P}^S$ satisfying efficiency and strategy-proofness.*

(P2.4) *There is no rule on $\mathcal{P}^S$ satisfying group rationality and efficiency.*

(P2.5) *There is no rule on $\mathcal{P}^S$ satisfying individual rationality and envy freeness on losses.*

(P2.6) *There is no rule on $\mathcal{P}^S$ satisfying individual rationality, efficiency, and envy freeness.*

Proof

(P2.1) It is obvious.

(P2.2) Let f be an efficient rule on $\mathcal{P}^S$. We prove that f satisfies (E2.1). Suppose not; *i.e.*, there exists $\succeq \in \mathcal{P}^S$ such that $\sum_{j\in N} f_j(\succeq) \neq [\sum_{j\in N} p_j]$. We proceed with the proof assuming that $\sum_{j\in N} f_j(\succeq) < [\sum_{j\in N} p_j]$ (the proof of the other case is analogous, and hence we omit it). Then, and since $\sum_{j\in N} f_j(\succeq) \in \mathbb{N}$, there exists at least one $j' \in N$ such that

$$f_{j'}(\succeq) < p_{j'}. \tag{3}$$

Moreover, we can assume that $f_i(\succeq) \leq p_i$ for all $i \in N$. To see that, suppose $f_{i'}(\succeq) > p_{i'}$ for some i'. Then, there would exist $\varepsilon > 0$ such that $p_{i'} < y_{i'} = f_{i'}(\succeq) - \varepsilon < f_{i'}(\succeq)$ and $f_{j'}(\succeq) < y_{j'} = f_{j'}(\succeq) + \varepsilon < p_{j'}$. Set $y_i = f_i(\succeq)$, for all $i \neq i', j'$. Hence, $\sum_{j\in N} y_j = \sum_{j\in N} f_j(\succeq) \in \mathbb{N}$. Thus, $y \in FA$ and $y_{i'} \succ_{i'} f_{i'}(\succeq)$, $y_{j'} \succ_{j'} f_{j'}(\succeq)$ and $y_i \sim_i f_i(\succeq)$ for all $i \neq i', j'$, which would imply that f is not efficient. By (3), $\sum_{j\in N} f_j(\succeq) < \sum_{j\in N} p_j$. If $\sum_{j\in N} f_j(\succeq) < [\sum_{j\in N} p_j] \leq \sum_{j\in N} p_j$, then there would exist $y \in FA$ such that $\sum_{j\in N} y_j = [\sum_{j\in N} p_j]$, $y_i \in [f_i(\succeq), p_i]$ for all i, and $y_{j'} \in (f_{j'}(\succeq), p_{j'}]$ for at least one j', contradicting efficiency of f. Thus, we can assume that $\sum_{j\in N} f_j(\succeq) < \sum_{j\in N} p_j < [\sum_{j\in N} p_j]$. Define $z = \sum_{j\in N} p_j - \sum_{j\in N} f_j(\succeq) > 0$. By definition of $[\sum_{j\in N} p_j]$, and since $\sum_{j\in N} f_j(\succeq) \in \mathbb{N}$, $\sum_{j\in N} p_j < [\sum_{j\in N} p_j] < \sum_{j\in N} p_j + z$. Define $y = (y_1, \ldots, y_n) \in \mathbb{R}^N_+$ such that $\sum_{j\in N} y_j \in \mathbb{N}$ and, for all $i \in N$, $y_i \in [p_i, 2p_i - f_i(\succeq)]$ and $y_{j'} \in (p_{j'}, 2p_{j'} - f_{j'}(\succeq))$ for at least one $j' \in N$. To see that there exists such y with the property that $\sum_{j\in N} y_j =$

$[\sum_{j\in N} p_j] \in \mathbb{N}$, observe that

$$\sum_{j\in N} p_j < \sum_{j\in N} y_j < \sum_{j\in N} 2p_j - \sum_{j\in N} f_j(\succeq) = \sum_{j\in N} p_j + z.$$

Since preferences are symmetric single-peaked, for all $i \in N$, $y_i \succeq_i f_i(\succeq)$ and there exists j' such that $y_{j'} \succ_{j'} f_{j'}(\succeq)$. Hence, f is not efficient. This proves that (E2.1) holds.

We now prove that f satisfies (E2.2). We only consider the case $\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j]$ (the proof of the other case is analogous, and hence we omit it). Suppose not. Then, there exists $i \in N$ such that $f_i(\succeq) > p_i$. Since, by hypothesis and (E2.1), $\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j] = \sum_{j\in N} f_j(\succeq)$, there exists $j' \in N$ such that $f_{j'}(\succeq) < p_{j'}$. Let ε be such that $0 < \varepsilon < \min\{f_i(\succeq) - p_i, p_{j'} - f_{j'}(\succeq)\}$. Then, the feasible vector of allotments $(f_i(\succeq) - \varepsilon, f_{j'}(\succeq) + \varepsilon, (f_j(\succeq))_{j\in N\setminus\{i,j'\}})$ Pareto dominates $f(\succeq)$. Hence, f is not efficient. This proves that (E2.2) holds.

We now prove the reciprocal. Let f be a rule satisfying (E2.1) and (E2.2). We only consider the case $\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j]$ (the proof of the other case is analogous, and hence we omit it). By (E2.2), $f_i(\succeq) \leq p_i$ for all $i \in N$. Suppose f is not efficient. Then, there exists $y = (y_1, ..., y_n) \in FA$ that Pareto dominates $f(\succeq)$. Since preferences are symmetric single-peaked, for all $i \in N$, $y_i \in [f_i(\succeq), p_i + (p_i - f_i(\succeq))]$ and $y_{j'} \in (f_{j'}(\succeq), p_{j'} + (p_{j'} - f_{j'}(\succeq)))$ for some $j' \in N$. By (E2.1), the definition of the integer $[\sum_{j\in N} p_j]$, the fact that $\sum_{j\in N}(p_j - f_j(\preceq)) \leq 0.5$ and our assumption,

$$\sum_{j\in N} f_j(\succeq) = [\sum_{j\in N} p_j] \leq \sum_{j\in N} p_j \leq [\sum_{j\in N} p_j] + 0.5$$

holds. Thus,

$$\begin{aligned}
\sum_{j\in N} f_j(\succeq) &< \sum_{j\in N} y_j \\
&< \sum_{j\in N} (p_j + (p_j - f_j(\succeq))) \\
&= \sum_{j\in N} p_j + \sum_{j\in N} (p_j - f_j(\succeq)) \\
&\leq \sum_{j\in N} p_j + 0.5 \\
&\leq [\sum_{j\in N} p_j] + 1.
\end{aligned}$$

In particular, $\sum_{j\in N} y_j < [\sum_{j\in N} p_j] + 1$. Since, by (E2.1), $\sum_{j\in N} f_j(\succeq) = [\sum_{j\in N} p_j]$ and $\sum_{j\in N} y_j \in \mathbb{N}$, we deduce that $\sum_{j\in N} y_j = [\sum_{j\in N} p_j] + 1$, a contradiction.

(P2.3) Assume f is efficient and strategy-proof on $\mathcal{P}^S$. We evaluate f at five problems $(N, \succeq^t) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $t = 1, 2, 3, 4$, and 5.

Consider the profile $\succeq^1$ where $p^1 = (0.26, 0.26)$. By (P2.2) in Proposition 2, $f_1(\succeq^1) + f_2(\succeq^1) = 1$ and $f_i(\succeq^1) \geq 0.26$ for all $i \in N$. Let $\succeq^2$ be such that $p^2 = (0.26, 0)$. By (P2.2) in Proposition 2, $f_1(\succeq^2) + f_2(\succeq^2) = 0$. Thus, $f(\succeq^2) = (0, 0)$. Let $\succeq^3$ be such that

$p^3 = (0, 0.26)$. Similarly, $f(\succeq^3) = (0, 0)$. By strategy-proofness, $f_1(\succeq^1) \succeq_1^1 f_1(\succeq^3) = 0$. Since preferences are symmetric, $f_1(\succeq^1) \leq 0.52$. Similarly, $f_2(\succeq^1) \leq 0.52$. Thus, $0.48 \leq f_i(\succeq^1) \leq 0.52$ for all $i \in N$.

Consider the profile $\succeq^4$ where $p^4 = (0.26, 0.3)$. Similarly to $\succeq^1$, we can prove that $0.4 \leq f_1(\succeq^4) \leq 0.52$ and $0.48 \leq f_2(\succeq^4) \leq 0.6$. We now obtain a contradiction in each of the three possible cases below.

1. $f_2(\succeq^1) > f_2(\succeq^4)$. Since $f_2(\succeq^4) \geq 0.48 > 0.26 = p_2^1$ and preferences are symmetric single-peaked, $f_2(\succeq^4) \succ_2^1 f_2(\succeq^1)$, which contradicts strategy-proofness because agent 2 manipulates f at profile $\succeq^1$ via $\succeq_2^4$ with $p_2^4 = 0.3$.

2. $f_2(\succeq^1) < f_2(\succeq^4)$. Since $f_2(\succeq^1) \geq 0.48 > 0.3 = p_2^4$ and preferences are symmetric single-peaked, $f_2(\succeq^1) \succ_2^4 f_2(\succeq^4)$, which contradicts strategy-proofness because agent 2 manipulates f at profile $\succeq^4$ via $\succeq_2^1$ with $p_2^1 = 0.26$.

3. $f_2(\succeq^1) = f_2(\succeq^4)$. Thus, $f_1(\succeq^1) = f_1(\succeq^4)$ and $0.48 \leq f_i(\succeq^4) \leq 0.52$ for all $i \in N$. Consider the profile $\succeq^5$ where $p^5 = (0.21, 0.3)$. Similarly to the profile $\succeq^1$ we can show that $0.4 \leq f_1(\succeq^5) \leq 0.42$ and $0.58 \leq f_2(\succeq^5) \leq 0.6$. Since $f_1(\succeq^4) = 0.48 > 0.42 \geq f_1(\succeq^5) > 0.26 = p_1^4$ and preferences are symmetric single-peaked, $f_1(\succeq^5) \succ_1^4 f_1(\succeq^4)$, which contradicts strategy-proofness because agent 1 manipulates f at profile $\succeq^4$ via $\succeq_1^5$ with $p_1^5 = 0.21$.

(P2.4) Assume f satisfies group rationality and efficiency on $\mathcal{P}^S$. Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2, 3\}$ and $p = (0.8, 0.4, 0.4)$. By efficiency $\sum_{i \in N} f_i(\succeq) = 2$ and $f_i(\succeq) \geq p_i$ for all $i \in N$. To apply the property of group rationality, consider the following table indicating, for each subset of agents with cardinality two, the aggregate loss, assuming the best integer amount is allotted (i.e., for each $S \subset N$ with $|S| = 2$, $\min_{k \in \mathbb{N}} \left| \sum_{j \in S} p_j - k \right|$).

S	$\min_{k \in \mathbb{N}} \left\| \sum_{j \in S} p_j - k \right\|$
$\{1, 2\}$	0.2
$\{1, 3\}$	0.2
$\{2, 3\}$	0.2

Observe that $0.4 = \left| \sum_{j \in N} p_j - \sum_{j \in N} f_j(\succeq) \right| = \sum_{j \in N} (f_j(\succeq) - p_j)$. Suppose first that $f_i(\succeq) - p_i = x$ for all $i \in N$. Then, $x = \frac{0.4}{3}$ and for any $S \subsetneq N$ with two agents, $\left| \sum_{j \in S} p_j - \sum_{j \in S} f_j(\succeq) \right| = \frac{0.8}{3} > 0.2 = \min_{x \in \mathbb{N}} \left| \sum_{j \in S} p_j - x \right|$. Hence, f does not satisfy group rationality. Suppose now that there exists $i \in N$ such that $(f_i(\succeq) - p_i) < \frac{0.4}{3}$.

Then, by setting $S = N \backslash \{i\}$, $\left|\sum_{j\in S} p_j - \sum_{j\in S} f_j(\succeq)\right| > \frac{0.8}{3} > 0.2 = \min_{k\in\mathbb{N}} \left|\sum_{j\in S} p_j - k\right|$, again a contradiction with group rationality of f.

(P2.5) Assume f satisfies individual rationality and envy freeness on losses on $\mathcal{P}^S$. Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (1, 0.7)$. By individual rationality, $f_1(\succeq) = 1$. Thus, $f_2(\succeq) \in \{0, 1, 2, \ldots\}$ which means that agent 2 envies the zero loss $(f_1(\succeq) - p_1 = 0)$ of agent 1.

(P2.6) Assume f satisfies individual rationality, efficiency, and envy-freeness on $\mathcal{P}^S$. Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (0.2, 0.35)$. By individual rationality, $0 \leq f_1(\succeq) \leq 0.4$ and $0 \leq f_2(\succeq) \leq 0.7$. By efficiency and (P2.2) in Proposition 2, $f_1(\succeq) + f_2(\succeq) = 1$. Thus, $0.3 \leq f_1(\succeq) \leq 0.4$ and $0.6 \leq f_2(\succeq) \leq 0.7$. Then, $f_1(\succeq) \succ_2 f_2(\succeq)$, which contradicts envy-freeness. ■

Our main objective in this paper is to identify individually rational rules to be used to solve the division problem when the integer number of units is endogenous and agents' preferences are symmetric single-peaked. Part (P2.1) in Proposition 2 characterizes the class of all individually rational rules. Since this class is large, it is natural to ask whether individual rationality is compatible with other additional properties. Efficiency and strategy-proofness emerge as two of the most basic and desirable properties. However, (P2.3) in Proposition 2 says that no rule satisfies the two properties simultaneously. In the next two subsections we study rules that are individually rational and efficient (Subsection 5.2) and rules that are individually rational and strategy-proof (Subsection 5.3). For the first case, we identify the constrained equal losses rule and the constrained equal awards rule as the unique ones that in addition of being individually rational and efficient satisfy also either justified envy-freeness on losses or envy-freeness on awards, respectively (Theorem 1). In contrast, in Subsection 5.3 we first show that although there are individually rational and strategy-proof rules, they are not very interesting (since they are not unanimous, for instance). Then, we show in Proposition 4 that individually rationality and strategy-proofness are indeed incompatible with unanimity.

5.2 Individual rationality and efficiency

Let $\succeq \in \mathcal{P}^S$ be a problem. Denote by $IRE(\succeq)$ the set of feasible vector of allotments satisfying individual rationality and efficiency. By (P2.1) and (P2.2) in Proposition 2, this set can be written as

$$\begin{aligned} IRE(\succeq) = \{x \in \mathbb{R}^N_+ \mid \quad & \textstyle\sum_{j\in N} x_j = [\sum_{j\in N} p_j] \text{ and, for all } i \in N, \\ & l_i \leq x_i \leq p_i \text{ when } \textstyle\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j] \text{ and} \\ & p_i \leq x_i \leq u_i \text{ when } \textstyle\sum_{j\in N} p_j < [\sum_{j\in N} p_j]\}. \end{aligned}$$

By Proposition 1, the set $IRE(\succeq)$ is always non-empty. Hence, a rule f satisfies individual rationality and efficiency if and only if, for each $\succeq \in \mathcal{P}^S$, $f(\succeq) \in IRE(\succeq)$.

However, individual rationality and efficiency are properties of rules that apply only to each problem separately. They do not impose conditions on how the rule should behave across problems. Thus, and given two different criteria compatible with individual rationality and efficiency, a rule can choose, in an arbitrary way, at problem $\succeq$ an allocation in $IRE(\succeq)$, following one criterion, while choosing at problem $\succeq'$ an allocation in $IRE(\succeq')$, following the other criterion. For instance the rule f that selects $f^{CEL}(\succeq)$ when $[\sum_{j\in N} p_j]$ is odd and $f^{CEA}(\succeq)$ when $[\sum_{j\in N} p_j]$ is even satisfies individual rationality and efficiency.[11] Thus, it seems appropriate to require that the rule satisfies an additional property in order to eliminate this kind of arbitrariness. We will focus on two alternative properties related to envy-freeness: justified envy-freeness on losses and envy-freeness on awards. But then, the consequence of requiring that rules (in addition of being individually rational and efficient) satisfy either one of these two forms of non-envyness are that only two rules are left, either the constrained equal losses or the constrained equal awards, respectively. But before stating in Theorem 1 below the characterizations of the two rules, we provide in Proposition 3 preliminary results on the two rules, that will be useful in the sequel.

Proposition 3

(P3.1) *The constrained equal losses rule on $\mathcal{P}^S$ satisfies individual rationality, efficiency, justified envy-freeness on losses, participation, unanimity and equal treatment of equals.*

(P3.2) *The constrained equal losses rule on $\mathcal{P}^S$ does not satisfy strategy-proofness, group rationality, envy-freeness, envy-freeness on losses, and envy-freeness on awards.*

(P3.3) *The constrained equal awards rule on $\mathcal{P}^S$ satisfies individual rationality, efficiency, envy-freeness on awards, participation, unanimity and equal treatment of equals.*

(P3.4) *The constrained equal awards rule on $\mathcal{P}^S$ does not satisfy strategy-proofness, group rationality, envy-freeness, envy-freeness on losses, and justified envy-freeness on losses.*

Proof

(P3.1) That f^{CEL} satisfies *unanimity* and *equal treatment of equals* follow directly from its definition. Now, we show that f^{CEL} satisfies the other properties.

Individual rationality. By its definition, for all $\succeq \in \mathcal{P}^S$ and $i \in N$, $f_i^{CEL}(\succeq) \in [l_i, u_i]$. By (P2.1) in Proposition 2, f^{CEL} is individually rational.

Efficiency. By its definition, f^{CEL} satisfies conditions (E2.1) and (E2.2) in Proposition 2. Hence, by (P2.2), f^{CEL} is efficient.

[11] Proposition 3 below will guarantee that for all $\succeq \in \mathcal{P}^S$, $f^{CEL}(\succeq), f^{CEA}(\succeq) \in IRE(\succeq)$.

Justified envy-freeness on losses. Suppose that $\sum_{ji \in N} p_j \geq [\sum_{j \in N} p_j]$ (the proof of the other case is analogous, and hence we omit it). Let $j \in N$ be such that $f_j^{CEL}(\succeq) \notin \mathbb{N}$. By individual rationality and single-peakedness,

$$f_j^{CEL}(\succeq) \succ_j k \text{ for all } k \in \mathbb{N}. \tag{4}$$

We want to show that for all $i \in N$, $f_i^{CEL}(\succeq) \succeq_i \max\{p_i + (f_j^{CEL}(\succeq) - p_j), 0\}$. By definition, $f_j^{CEL}(\succeq) = p_j - \min\{\widehat{\alpha}, p_j - l_j\}$. If $p_j - l_j \leq \widehat{\alpha}$, then $f_j^{CEL}(\succeq) = l_j$, which contradicts (4) because $f_j^{CEL}(\succeq) \sim_j l_j \sim u_j$ and either l_j or u_j is an integer. Hence,

$$f_j^{CEL}(\succeq) = p_j - \widehat{\alpha}. \tag{5}$$

Let $i \in N$ be arbitrary. We distinguish between two cases. First, $\widehat{\alpha} \leq p_i - l_i$. Then, by (5), $f_i^{CEL}(\succeq) = p_i - \widehat{\alpha} = p_i + (f_j^{CEL}(\succeq) - p_j)$, which means that $f_i^{CEL}(\succeq) = \max\{p_i + (f_j^{CEL}(\succeq) - p_j), 0\}$. Hence, $f_i^{CEL}(\succeq) \succeq_i \max\{p_i + (f_j^{CEL}(\succeq) - p_j), 0\}$. Second, $\widehat{\alpha} > p_i - l_i$. Then, by definition, $f_i^{CEL}(\succeq) = l_i$. Since, by (5), $p_i + (f_j^{CEL}(\succeq) - p_j) = p_i - \widehat{\alpha} < l_i \leq p_i$, single-peakedness implies that $f_i^{CEL}(\succeq) \succeq_i \max\{p_i + (f_j^{CEL}(\succeq) - p_j), 0\}$.

Participation. Let $\succeq \in \mathcal{P}^S$ be such that $k \leq p_i$ for some $i \in N$ and $k \in \mathbb{N}$. We want to show that $f_i^{CEL}(\succeq) = k + f_i^{CEL}\left(\succeq_i^{-k}, \succeq_{-i}\right)$. Set $\succeq' = (\succeq_i^{-k}, \succeq_{-i})$ and $p' = (p_i - k, (p_j)_{j \in N \setminus \{i\}})$. Suppose that $\sum_{j \in N} p_j \geq [\sum_{j \in N} p_j]$ (the proof of the other case is analogous, and hence we omit it). Then, $f_i^{CEL}(\succeq) = p_i - \min\{\widehat{\alpha}, p_i - l_i\}$ where $\widehat{\alpha}$ satisfies $\sum_{j \in N} f_j^{CEL}(\succeq) = [\sum_{j \in N} p_j]$. Since $p'_i = p_i - k$ and k is an integer, $[\sum_{j \in N} p'_j] = [\sum_{j \in N} p_j] - k$. Hence, $\sum_{j \in N} p'_j \geq [\sum_{j \in N} p'_j]$. Now, $f_i^{CEL}(\succeq') = p'_i - \min\{\widehat{\alpha}', p'_i - l'_i\}$ where $\widehat{\alpha}'$ satisfies $\sum_{j \in N} f_j^{CEL}(\succeq') = [\sum_{j \in N} p'_j]$. Since $l'_i = l_i - k$ and $l'_j = l_j$ for all $j \in N \setminus \{i\}$, we deduce that $\widehat{\alpha}' = \widehat{\alpha}$. Then,

$$\begin{aligned} f_i^{CEL}(\succeq') &= p_i - k - \min\{\widehat{\alpha}, p_i - k - (l_i - k)\} \\ &= p_i - \min\{\widehat{\alpha}, p_i - l_i\} - k \\ &= f_i^{CEL}(\succeq) - k, \end{aligned}$$

which is what we wanted to show.

(P3.2) We show that f^{CEL} does not satisfy the following properties on $\mathcal{P}^S$.

Strategy-proofness. Consider the problems $(N, \succeq)$ and $(N, \succeq')$ where $N = \{1, 2\}$, $p = (0.4, 0.8)$ and $p' = (0.4, 0.9)$. Then, $f^{CEL}(\succeq) = (0.3, 0.7)$ and $f^{CEL}(\succeq') = (0.25, 0.75)$. Since $0.75 \succ_2 0.7$, f^{CEL} does not satisfy strategy-proofness because agent 2 manipulates f^{CEL} at profile $\succeq$ via $\succeq'_2$.

Group rationality. It follows from (P3.1) and (P2.4).

Envy-freeness. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.40, 0.46)$. Then, $f^{CEL}(\succeq) = (0.47, 0.53)$, which contradicts envy-freeness because agent 2 strictly prefers 0.47 to 0.53.

Envy-freeness on losses. If follows from (P3.1) and (P2.6).

Envy-freeness on awards. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.4, 0.46)$. Then, $f^{CEL}(\succeq) = (0.47, 0.53)$. Therefore, $a_1(\succeq, f^{CEL}) = 0.8 - 0.47 = 0.33$ and $a_2(\succeq, f^{CEL}) = 0.92 - 0.53 = 0.39$. For $0.38 \in [0.33, 0.39]$, we have that $f_1^{CEL}(\succeq) = 0.47 \prec_1 0.38$. Thus, f^{CEL} does not satisfy envy-freeness on awards.

(P3.3) That f^{CEA} satisfies *unanimity* and *equal treatment of equals* follow directly from its definition. Now, we show that f^{CEA} satisfies the other properties.

Individual rationality. By its definition, for all $\succeq \in \mathcal{P}^S$ and $i \in N$, $f_i^{CEA}(\succeq) \in [l_i, u_i]$. By (P2.1) in Proposition 2, f^{CEL} is individually rational.

Efficiency. By its definition, f^{CEA} satisfies conditions (E2.1) and (E2.2) in Proposition 2. Hence, by (P2.2), f^{CEA} is efficient.

Envy-freeness on awards. Suppose that $\sum_{j \in N} p_j \geq [\sum_{j \in N} p_j]$ (the proof of the other case is analogous, and hence we omit it). Since f^{CEA} is efficient, by (E2.2) in (P2.2) of Proposition 2, $f_i^{CEA}(\succeq) \leq p_i$ for all $i \in N$. Suppose that f^{CEA} does not satisfy envy-freeness on awards. Then, there exist $i, j \in N$ and $x \in [\min\{a_i, a_j\}, \max\{a_i, a_j\}]$ such that

$$l_i + x \succ_i f_i^{CEA}(\succeq). \tag{6}$$

Since $f_i^{CEA}(\succeq) \leq p_i$, the allotment $f_i^{CEA}(\succeq)$ is not the peak of $\succeq_i$ and so $f_i^{CEA}(\succeq) < p_i$. Moreover, since by definition $f_i^{CEA}(\succeq) = l_i + \min\{\widehat{\beta}, p_i - l_i\}$, $\widehat{\beta} < p_i - l_i$ and

$$f_i^{CEA}(\succeq) = l_i + \widehat{\beta} \tag{7}$$

hold. Thus, $a_i = \widehat{\beta}$. We distinguish between two cases. First, $\min\{\widehat{\beta}, p_j - l_j\} = \widehat{\beta}$. Since $a_j = f_j^{CEA}(\succeq) - l_j = \widehat{\beta}$, it must be the case that $x = \widehat{\beta}$. Hence, by (6),

$$l_i + \widehat{\beta} = l_i + x \succ_i f_i^{CEA}(\succeq) = l_i + \widehat{\beta},$$

which is a contradiction. Second, $\min\{\widehat{\beta}, p_j - l_j\} = p_j - l_j < \widehat{\beta}$. By the definition of f^{CEA}, $f_j^{CEA}(\succeq) = p_j$ and $a_j = f_j^{CEA}(\succeq) - l_j = p_j - l_j$. Thus, $x \in [p_j - l_j, \widehat{\beta}]$ and

$$l_i + x \leq l_i + \widehat{\beta} = f_i^{CEA}(\succeq) \leq p_i,$$

where the equality follows from (7). By single-peakedness, $f_i^{CEA}(\succeq) \succeq_i l_i + x$, a contradiction with (6).

Participation. Let $\succeq \in \mathcal{P}^S$ be such that $k \leq p_i$ for some $i \in N$ and $k \in \mathbb{N}$. We want to show that $f_i^{CEA}(\succeq) = k + f_i^{CEA}\left(\succeq_i^{-k}, \succeq_{-i}\right)$. Set $\succeq' = (\succeq_i^{-k}, \succeq_{-i})$ and $p' = (p_i - k, (p_j)_{j \in N \setminus \{i\}})$. Suppose that $\sum_{j \in N} p_j \geq [\sum_{j \in N} p_j]$ (the proof of the other case is analogous, and hence we omit it). Then, $f_i^{CEA}(\succeq) = l_i + \min\{\widehat{\beta}, p_i - l_i\}$ where $\widehat{\beta}$ satisfies $\sum_{j \in N} f_j^{CEA}(\succeq) = [\sum_{j \in N} p_j]$. Since $p'_i = p_i - k$ and k is an integer, $[\sum_{j \in N} p'_j] = [\sum_{j \in N} p_j] - k$. Hence, $\sum_{j \in N} p'_j \geq [\sum_{j \in N} p'_j]$. Now, $f_i^{CEA}(\succeq') = l'_i + \min\{\widehat{\beta}', p'_i - l'_i\}$ where $\widehat{\beta}'$ satisfies $\sum_{j \in N} f_j^{CEA}(\succeq') = [\sum_{j \in N} p'_j]$. Since $l'_i = l_i - k$ and $l'_j = l_j$ for all $j \in N \setminus \{i\}$, we deduce that $\widehat{\beta}' = \widehat{\beta}$. Then,

$$\begin{aligned} f_i^{CEA}(\succeq') &= l_i - k + \min\{\widehat{\beta}, p_i - k - (l_i - k)\} \\ &= l_i + \min\{\widehat{\beta}, p_i - l_i\} - k \\ &= f_i^{CEA}(\succeq) - k, \end{aligned}$$

which is what we wanted to prove.

(P3.4) We show that f^{CEA} does not satisfy the following properties on $\mathcal{P}^S$.

Strategy-proofness. Consider the problems $(N, \succeq)$ and $(N, \succeq')$ where $N = \{1, 2\}$, $p = (0.4, 0.8)$ and $p' = (0.6, 0.8)$. Then, $f^{CEA}(\succeq) = (0.2, 0.8)$ and $f^{CEA}(\succeq') = (0.3, 0.7)$. Since $0.3 \succ_1 0.2$, f^{CEA} does not satisfy strategy-proofness because agent 1 manipulates f^{CEA} at profile $\succeq$ via $\succeq'_1$.

Group rationality. It follows from (P3.3) and (P2.4).

Envy-freeness. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.6, 0.8)$. Then, $f^{CEA}(\succeq) = (0.3, 0.7)$, which means that f^{CEA} is not envy-free because agent 1 strictly prefers 0.7 to 0.3.

Envy-freeness on losses. It follows from (P3.3) and (P2.5).

Justified envy-freeness on losses. Consider the problem $(N, \succeq)$ where $N = \{1, 2\}$ and $p = (0.6, 0.8)$. Then, $f^{CEA}(\succeq) = (0.3, 0.7)$, which means that f^{CEA} does not satisfy justified envy-freeness on losses because agent 1 strictly prefers $0.6 + (0.7 - 0.8) = 0.5$ to 0.3. ■

Theorem 1, the main result of the paper, characterizes axiomatically the constrained equal losses rule f^{CEL} and the constrained equal awards rule f^{CEA} on the domain of symmetric single-peaked preferences.

Theorem 1 *The following two characterizations hold.*

(T1.1) *The constrained equal losses rule f^{CEL} is the unique rule on $\mathcal{P}^S$ satisfying individual rationality, efficiency, and justified envy-freeness on losses.*

(T1.2) *The constrained equal awards rule f^{CEA} is the unique rule on $\mathcal{P}^S$ satisfying individual rationality, efficiency, and envy-freeness on awards.*

Proof By Proposition 3, f^{CEL} satisfies individual rationality, efficiency and justified envy-freeness on losses and f^{CEA} satisfies individual rationality, efficiency and envy-freeness on awards.

Before we prove uniqueness for each rule separately, let $\succeq \in \mathcal{P}^S$ be a problem and let f be a rule satisfying individual rationality and efficiency. Suppose that $\sum_{j\in N} p_j \geq [\sum_{j\in N} p_j]$ (the proof for the other case is analogous, and hence we omit it). Since f is efficient, by (E2.1) and (E2.2) in (P2.2) of Proposition 2, $\sum_{j\in N} f_j(\succeq) = [\sum_{j\in N} p_j]$ and

$$f_i(\succeq) \leq p_i \tag{8}$$

for all $i \in N$. By individual rationality and (P2.1) in Proposition 2, $f_i(\succeq) \geq l_i$ for all $i \in N$.

We first show uniqueness of f^{CEL}. For each $i \in N$, $f_i(\succeq) = p_i - x_i$ where $0 \leq x_i \leq p_i - l_i$. Assume that $x_j < x_i$ for some pair $i, j \in N$. By single peakedness, $p_i - x_j \succ_i p_i - x_i$. Since

$$f_i(\succeq) = p_i - x_i \prec_i p_i - x_j = p_i + (f_j(\succeq) - p_j)$$

holds, by justified envy-freeness on losses, there must exist $y_j \in \mathbb{N}$ such that $f_j(\succeq) \simeq y_j$. By individual rationality,

$$f_j(\succeq) = l_j. \tag{9}$$

Let S be the set of agents with the largest loss from the peak. Namely, $S = \{i' \in N \mid x_{i'} \geq x_{j'} \text{ for all } j' \in N\}$. Since N is finite, $S \neq \emptyset$. If $S = N$, then there exists x such that $x \in [0, p_i - l_i]$ and $f_i(\succeq) = p_i - x$ for all $i \in N$. Set $\widehat{\alpha} = x$. Assume $S \subsetneq N$. For all $j, j' \in S$, $x_j = x_{j'}$. Set $\widehat{\alpha} = x_j$ and observe that $f_j(\succeq) = p_j - \widehat{\alpha} \geq p_j - l_j$. If $j \notin S$, then there exists $i \in S$ such that $x_j < x_i$. By (9), $f_j(\succeq) = l_j$. Since $\widehat{\alpha} > x_j$ for all $j \notin S$, $f_j(\succeq) = l_j = p_j - \min\{\widehat{\alpha}, p_j - l_j\}$. Thus, $f(\succeq) = f^{CEL}(\succeq)$.

We now show uniqueness of f^{CEA}. By (8), for each $i \in N$, $f_i(\succeq) = l_i + a_i$ and $0 \leq a_i \leq p_i - l_i$.

We first prove that if $a_i < a_j$ for some $i, j \in N$, then $a_i = p_i - l_i$. Assume not; there exist $i, j \in N$ such that $a_i < a_j$ and $a_i < p_i - l_i$. Let $x \in \mathbb{R}_+$ be such that $x \in (a_i, \min\{a_j, p_i - l_i\}]$. Since $f_i(\succeq) = l_i + a_i < l_i + x < p_i$, single-peakedness implies that $l_i + x \succ_i f_i(\succeq)$ where $x \in (a_i, a_j]$, contradicting envy-freeness on awards.

Let S be the set of agents with the largest award from the peak. Namely, $S = \{i' \in N \mid a_{i'} \geq a_{j'} \text{ for all } j' \in N\}$. Since N is finite, $S \neq \emptyset$. If $S = N$, then there exists a such that $a \in [0, p_i - l_i]$ and $f_i(\succeq) = p_i + a$ for all $i \in N$. Set $\widehat{\beta} = a$. Assume $S \subsetneq N$. For all $j, j' \in S$, $a_j = a_{j'}$. Set $\widehat{\beta} = a_j$ and observe that $f_j(\succeq) = l_j + \widehat{\beta} \leq p_j - l_j$. If $j \notin S$, then there exists $i \in S$ such that $a_i > a_j$. Then, $a_j = p_j - l_j$. Hence, $f_j(\succeq) = l_j + a_j = l_j + p_j - l_j = p_j$ and $p_j - l_j \leq \widehat{\beta}$. Thus, for all $j \in N$, $f_j(\succeq) = f_j^{CEA}(\succeq)$. ■

Remark 2 The two sets of properties used in the two characterizations of Theorem 1 are independent.

(R2.1) The rule f defined by setting $f_i(\succeq) = [p_i]$ for all $\succeq \in \mathcal{P}^S$ and all $i \in N$ satisfies individual rationality and justified envy-freeness on losses but it is not efficient.

(R2.2) The rule f^{EL} satisfies efficiency and justified envy-freeness on losses but is not individually rational.

(R2.3) The rule f^{CEA} satisfies individual rationality and efficiency but it does not satisfy justified envy-freeness on losses.

(R2.4) The rule f defined by setting $f_i(\succeq) = [p_i]$ for all $\succeq \in \mathcal{P}^S$ and all $i \in N$ satisfies individual rationality and envy-freeness on awards but it is not efficient.

(R2.5) The rule f^{EA} satisfies efficiency and envy-freeness on awards but it is not individually rational.

(R2.6) The rule f^{CEL} satisfies individual rationality and efficiency but it is not envy-freeness on awards.

5.3 Individual rationality and strategy-proofness

We now study the set of rules satisfying individual rationality and strategy-proofness on the set of symmetric single-peaked preferences. There are many rules satisfying both properties. For instance, the rule that selects $f(\succeq) = ([p_i])_{i \in N}$ for all $\succeq \in \mathcal{P}^S$ is individually rational and strategy-proof. But there are many more, yet some of them are very difficult to justify as reasonable solutions to the problem. Consider the following family of rules. For each vector $x \in \mathbb{R}^N_+$ satisfying $\sum_{i \in N} x_i \in \mathbb{N}$, define f^x as the rule that when x is at least as good as $([p_i])_{i \in N}$ for each $i \in N$, f^x selects x. Otherwise f^x selects $([p_i])_{i \in N}$. Formally, fix $x \in \mathbb{R}^N_+$ satisfying $\sum_{i \in N} x_i \in \mathbb{N}$. For each problem $\succeq \in \mathcal{P}^S$, set

$$f^x(\succeq) = \begin{cases} x & \text{if } x_i \succeq_i [p_i] \text{ for all } i \in N \\ ([p_i])_{i \in N} & \text{otherwise.} \end{cases}$$

It is easy to see that each rule in the family $\{f^x \mid x \in \mathbb{R}^N_+ \text{ and } \sum_{i \in N} x_i \in \mathbb{N}\}$ is individually rational and strategy-proof. However, they are arbitrary and non-interesting. Thus, we ask whether it is possible to identify a subset of individually rational and strategy-proof rules satisfying additionally a basic, weak and desirable property. We interpret Proposition 4 below as giving a negative answer to this question: individual rationality and strategy-proofness are not compatible even with unanimity, a very weak form of efficiency.

Proposition 4 *There is no rule on $\mathcal{P}^S$ satisfying individual rationality, strategy-proofness and unanimity.*

Proof To obtain a contradiction, assume that f is a rule satisfying individual rationality, strategy-proofness and unanimity. Consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (0.2, 0.8)$. Since preference are symmetric single-peaked we identify, in the remainder of this proof, each preference $\succeq_i$ with its peak p_i, and so we write $f(p)$ instead of $f(\succeq)$. By unanimity, $f(0.2, 0.8) = (0.2, 0.8)$. Consider $f(0.2, 0.5)$ and suppose that $f_2(0.2, 0.5) > 0.8$; then, agent 2 manipulates f at profile $(0.2, 0.5)$ via 0.8. This contradicts strategy-proofness of f. Hence, $f_2(0.2, 0.5) \leq 0.8$.

Claim: $f_2(0.2, 0.5) = 0.8$.

Proof: Suppose $f_2(0.2, 0.5) < 0.8$. Thus, $f(0.2, 0.5) = (0.2 + x, 0.8 - x)$ where $0 < x < 0.8$. By individual rationality of agent 1, $0 \leq 0.2 + x \leq 0.4$, which means that $x \leq 0.2$. By individual rationality of agent 2, $0 \leq 0.8 - x \leq 1$, which means that $x \leq 0.8$, which is not binding. Thus, $0 < x \leq 0.2$. Let $y \in \mathbb{R}_+$ be such that

$$0.2 - x \leq y < 0.2. \tag{10}$$

Thus, $f_1(y, 0.5) \leq 0.2 + x$ (otherwise agent 1 manipulates f at profile $(y, 0.5)$ via 0.2). To show that indeed $f_1(y, 0.5) = 0.2 + x$ we distinguish between two different cases:

1. $0.2 - x < f_1(y, 0.5) < 0.2 + x$. Then, agent 1 manipulates f at profile $(0.2, 0.5)$ via y. This contradicts strategy-proofness of f.

2. $f_1(y, 0.5) \leq 0.2 - x$. Since f satisfies individual rationality two subcases are possible.

   (a) $f_1(y, 0.5) + f_2(y, 0.5) = 1$. Then, $f_2(y, 0.5) \geq 0.8 + x$. By unanimity, $f_2(y, 1 - y) = 1 - y$. Thus, agent 2 manipulates f at profile $(y, 0.5)$ via $1 - y$ because $0.5 < 1 - y < 0.8 + x$, where the two inequalities follow from (10). This contradicts strategy-proofness of f.

   (b) $f_1(y, 0.5) + f_2(y, 0.5) = 0$. Then, $f_2(y, 0.5) = 0$. Again, agent 2 manipulates f at profile $(y, 0.5)$ via $1 - y$. This contradicts strategy-proofness of f.

Hence, $f_1(y, 0.5) = 0.2 + x$. We show now that $f_1(0.2 - x, 0.5) = 0.2 + x$. If $f_1(0.2 - x, 0.5) > 0.2 + x$ then 1 manipulates f at profile $(0.2 - x, 0.5)$ via y. Suppose $f_1(0.2 - x, 0.5) =: z < 0.2 + x$. If $z = y$, then agent 1 manipulates f at profile $(y, 0.5)$ via $0.2 - x$. If $z > y$, then agent 1 manipulates f at $(y, 0.5)$ via $0.2 - x$, because $|y - z| < |y - 0.2 - x|$ since $z - y < 0.2 + x - y$ if and only if $z < 0.2 + x$. If $z < y$, then agent 1 would manipulate f at $(y, 0.5)$ via $0.2 - x$, provided that $|y - z| < |y - 0.2 - x|$. But since $y - z < 0.2 + x - y$ if and only if (i) $2y - 0.2 - x < z$ and (ii) $z < 0.2 + x$; but (i) and (ii) hold since $2y - 0.2 - x < z < 0.2 + x$ holds because $y < 0.2 < 0.2 + x$. Hence, $f_1(0.2 - x, 0.5) = 0.2 + x$. Now, by individual rationality of agent 1, $|0.2 - x - 0| \geq |0.2 - x - 0.2 - x|$, so $0.2 - x \geq 2x$, or equivalently, $x \leq \frac{0.2}{3}$. Consider now the profile $(0.2 - x, 0.5)$ instead of $(0.2, 0.5)$. Since $f_1(0.2 - x, 0.5) = 0.2 - x + 2x$, applying the same argument as for the profile $(0.2, 0.5)$ we obtain that $f_1(0.2 - 3x, 5) = 0.2 + x$. By individual rationality of agent 1, $|0.2 - 3x - 0| \geq |0.2 - 3x - 0.2 - x|$, so $0.2 - 3x \geq 4x$, or equivalently, $x \leq \frac{0.2}{7}$. Since $x > 0$ and it is fixed, repeating this process several times we will eventually find a contradiction with individual rationality of agent 1. Then, $f(0.2, 0.5) = (0.2, 0.8)$, which proves the claim. □

Consider now the profile $(0.2, 0.39)$. We distinguish among three different cases:

1. $f_1(0.2, 0.39) + f_2(0.2, 0.39) \geq 2$. By individual rationality, $f_1(0.2, 0.39) \leq 0.4$ and $f_2(0.2, 0.39) \leq 0.78$, which is a contradiction.

2. $f_1(0.2, 0.39) + f_2(0.2, 0.39) = 1$. By individual rationality of agent 1, $f_1(0.2, 0.39) \leq 0.4$, and so $0.6 \leq f_2(0.2, 0.39)$. By individual rationality of agent 2, $f_2(0.2, 0.39) \leq 0.78$. Thus, agent 2 manipulates f at profile $(0.2, 0.5)$ via 0.39. This contradicts strategy-proofness.

3. $f_1(0.2, 0.39) + f_2(0.2, 0.39) = 0$. Then, $f_1(0.2, 0.39) = f_2(0.2, 0.39) = 0$. Similarly to the case of profile $(0.2, 0.5)$, we can prove that $f(0.38, 0.39) = (0.38, 0.62)$. Thus, agent 1 manipulates f at profile $(0.2, 0.39)$ via 0.38. This contradicts strategy-proofness.

Since we have obtained a contradiction in each of the possible cases, there does not exist a rule satisfying simultaneously the properties of individual rationality, strategy-proofness and unanimity. ■

By (R1.1) in Remark 1 and Proposition 4 we obtain the following Corollary.

Corollary 2 *There is no rule on $\mathcal{P}^S$ satisfying individual rationality, strategy-proofness and efficiency.*

6 Final remarks

Before finishing the paper we deal with two natural questions. First, are our results generalizable to rules defined on $\mathcal{P}$, the set of problems where agents have single-peaked preferences? Second, how do well-known rules, used to solve the division problem with a fixed amount of the good, behave when the number of units to allot is endogenous? We partially answer the two questions separately in each of the next two subsections.

6.1 Results for general single-peaked preferences

Obviously, all the impossibility results we have obtained for rules operating on the domain of symmetric single-peaked preferences also hold when they operate in the larger domain.

Proposition 5 contains some results on rules operating on the full domain of single-peaked preferences. But before stating it, we need some additional notation to refer to the extremes of the individually rational intervals for those preferences. Let $\succeq_i$ be a single-peaked preference with peak p_i. Define

$$b_i = \begin{cases} [p_i]_l & \text{if } [p_i]_l \succeq_i [p_i]_u \\ [p_i]_u & \text{otherwise.} \end{cases} \tag{11}$$

By continuity and single-peakedness, there are two numbers $\widehat{l}_i, \widehat{u}_i \in \mathbb{R}_+$ satisfying the following conditions: (i) $b_i \in \{\widehat{l}_i, \widehat{u}_i\}$; (ii) $\widehat{l}_i \sim \widehat{u}_i$; (iii) for each $y_i \in [\widehat{l}_i, \widehat{u}_i]$, $y_i \succeq_i b_i$; and (iv) for all $y_i \notin [\widehat{l}_i, \widehat{u}_i]$, $b_i \succ_i y_i$.

Proposition 5 *The following statements hold.*

(P5.1) *A rule f on $\mathcal{P}$ is individually rational if and only if, for all $\succeq \in \mathcal{P}$ and $i \in N$, $f_i(\succeq) \in [\widehat{l}_i, \widehat{u}_i]$.*

(P5.2) *A rule f on $\mathcal{P}$ is efficient if and only if, for all $\succeq \in \mathcal{P}$, two conditions hold:*

(E5.1) $\sum_{j \in N} f_j(\succeq) \in \{[\sum_{j \in N} p_j]_l, [\sum_{j \in N} p_j]_u\}$.

(E5.2) *For all $i \in N$, $f_i(\succeq) \leq p_i$ when $\sum_{j \in N} p_j \geq \sum_{j \in N} f_j(\succeq)$ and $f_i(\succeq) \geq p_i$ when $\sum_{j \in N} p_j < \sum_{j \in N} f_j(\succeq)$.*

(P5.3) *There exist rules on $\mathcal{P}$ satisfying individual rationality and efficiency.*

(P5.4) *There exist rules on $\mathcal{P}$ satisfying individual rationality and strategy-proofness.*

Proof (P5.1) It is obvious.

(P5.2) It is similar to the proof of (P2.2) in Proposition 2, and hence we omit it.

(P5.3) We show that for each $\succeq \in \mathcal{P}$ the set of individually rational and efficient vector of allotments is non-empty. Fix a profile $\succeq \in \mathcal{P}$ and let $p = (p_1, \ldots, p_n)$ be its associated vector of peaks. Similarly to (E5.1), any efficient vector of allotments y satisfies $\sum_{j \in N} y_j \in \{[\sum_{j \in N} p_j]_l, [\sum_{j \in N} p_j]_u\}$. For each $i \in N$, consider b_i defined as in (11). Suppose that $\sum_{j \in N} b_j \leq [\sum_{j \in N} p_j]_l$ (the other case is analogous, and hence we omit it). Consider the allotment $x = (x_1, \ldots, x_n)$ such that, for each $i \in N$,

$$x_i = \begin{cases} p_i & \text{if } b_i = [p_i]_l \\ \widehat{l}_i + \min\{p_i - \widehat{l}_i, \gamma\} & \text{otherwise,} \end{cases}$$

where γ satisfies $\sum_{j \in N} x_j = [\sum_{j \in N} p_j]_l$. By continuity of the vector $x = (x_1, \ldots, x_n)$ with respect to the parameter γ, and the fact that $\sum_{j \in N} b_j \leq [\sum_{j \in N} p_j]_l$ holds, such γ exists. By definition, $x_i \in [\widehat{l}_i, p_i]$ for all $i \in N$ and so, x_i is individually rational. Suppose that x is not efficient. By the definition of x, the Pareto improvement requires a different integer choice. Hence, there exists $y \in \mathbb{R}^N_+$ such that $\sum_{j \in N} y_j = [\sum_{j \in N} p_j]_u$ and $y_i \succeq_i x_i$ for all $i \in N$. Thus, y is a feasible vector of allotments satisfying individual rationality and efficiency.

(P5.4) Consider the rule f that, for each $\succeq \in \mathcal{P}$ and each $i \in N$, $f_i(\succeq) = b_i$, where b_i is defined as in (11). It is immediate to see that f is individually rational and strategy-proof. ■

In this case f^{CEL} and f^{CEA} are not efficient, as next example shows.

Example 3 Consider the problem $(N, \succeq) \in \mathcal{P}$ where $N = \{1, 2, 3\}$ and $p = (0.15, 0.5, 0.65)$. Thus, $f^{CEL}(N, \succeq) = (0.05, 0.4, 0.55)$ and $f^{CEA}(N, \succeq) = (0.15, 0.27, 0.57)$. If we take $y = (0.15, 0.9, 0.95)$ and $\succeq$ such that $0.9 \succ_2 0.4$ and $0.95 \succ_3 0.57$ we have that f^{CEL} and f^{CEA} are not efficient. □

6.2 Other rules

In the classical division problem, where a fixed amount of the good has to be allotted, the uniform rule emerges as the one that satisfies many desirable properties. For instance, Sprumont (1991) shows that it is the unique rule satisfying strategy-proofness, efficiency and anonymity. Sprumont (1991) also shows that in this characterization anonymity may be replaced by non-envyness and Ching (1994) shows that in fact anonymity may be replaced by the weaker requirement of equal treatment of equals. Sönmez (1994) shows that the uniform rule is the unique one satisfying consistency, monotonicity and individual rationality from equal division. Thomson (1994a, 1994b, 1995 and 1997) contains alternative characterizations of the uniform rule using the properties of one sided resource-monotonicity, converse consistency, weak population-monotonicity and replication invariance, respectively. On the other hand, if one is concerned mostly with incentives and efficiency issues (and leaves aside any equity principle), sequential dictator rules emerge as natural ways of solving the classical division problem, since they are strategy-proof and efficient. However, we briefly argue below that the natural adaptations of all these rules to our setting with endogenous integer units of the good are far from desirable since they are neither individually rational nor strategy-proof.

6.2.1 Uniform rule

We adapt the uniform rule by selecting the efficient number of integer units (using our tie-breaking criterion whenever there are two of them) and allotting this amount using the uniform rule.

Extended uniform (f^{EU}). For all $\succeq \in \mathcal{P}$ and $i \in N$, set

$$f_i^{EU}(\succeq) = \begin{cases} \max\{p_i, \eta\} & \text{if } \sum_{j\in N} p_j \geq [\sum_{j\in N} p_j] \\ \min\{p_i, \eta\} & \text{if } \sum_{j\in N} p_j < [\sum_{j\in N} p_j], \end{cases}$$

where η is the unique real number for which $\sum_{j\in N} f_j^{EU}(\succeq) = [\sum_{j\in N} p_j]$ holds.

Proposition 6 *The extended uniform rule f^{EU} is efficient on $\mathcal{P}$ (and hence on $\mathcal{P}^S$) but it is neither individually rational nor strategy-proof on $\mathcal{P}^S$ (and hence, on $\mathcal{P}$).*

Proof The same argument used to prove (E5.1) and (E5.2) on $\mathcal{P}$ shows that f^{EU} is efficient on $\mathcal{P}$ (and hence on $\mathcal{P}^S$). To see that f^{EU} is neither individually rational nor strategy-proof on $\mathcal{P}^S$ consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2, 3\}$ and $p = (0.2, 0.2, 0.9)$. Then $f^{EU}(\succeq) = (0.2, 0.2, 0.6)$. Since agent 3 strictly prefers 1 to 0.6, f^{EU} is not individually rational. To see that f^{EU} is not strategy-proof consider the symmetric single-peaked preference $\succeq_3'$ with $p_3' = 1.12$. Then, $f^{EU}(\succeq_3', \succeq_{-3}) = (0.44, 0.44, 1.12)$. Since 3 strictly prefers (according to $\succeq_3$) 1.12 to 0.6, agent 3 manipulates f^{EU} at profile $\succeq$ via $\succeq_3'$. ■

6.2.2 Sequential dictator

We adapt the sequential dictator rule to the setting where the integer number of units to be allotted is endogenous. Fix an ordering on the set of agents and let them select sequentially, following the ordering, the amount they want (their peak) among the set of all efficient allocations. Formally, let $\sigma : N \to \{1, \ldots, n\}$ be a one-to-one mapping defining an ordering on the set of agents N; namely, for $i, j \in N$, $\sigma(i) < \sigma(j)$ means that i goes before j in the ordering σ.

Sequential dictator at σ ($f^{SD\sigma}$). For all $\succeq \in \mathcal{P}^S$ and $i \in N$, set

$$f_i^{SD\sigma}(\succeq) = \begin{cases} \min\{p_i, \max\{[\sum\limits_{k\in N} p_k] - \sum\limits_{\{j'\in S|\sigma(j')<\sigma(i)\}} p_{j'}, 0\}\} & \text{if } \sigma(i) < \sigma(j) \text{ for some } j \\ \\ \max\{[\sum\limits_{k\in N} p_k] - \sum\limits_{\{j'\in S|\sigma(j')<\sigma(i)\}} p_{j'}, 0\} & \text{otherwise.} \end{cases}$$

Proposition 7 *The sequential dictator rule $f^{SD\sigma}$ at any ordering σ is efficient on $\mathcal{P}^S$ but it is neither individually rational nor strategy-proof on $\mathcal{P}^S$.*

Proof The fact that, for any fixed ordering σ, $f^{SD\sigma}$ is efficient on $\mathcal{P}$ follows immediately from its definition and (P5.2) in Proposition 5. To see that $f^{SD\sigma}$ is neither individually rational nor strategy-proof on $\mathcal{P}^S$ consider the problem $(N, \succeq) \in \mathcal{P}^S$ where $N = \{1, 2\}$ and $p = (0.26, 0.26)$. Without loss of generality, let $\sigma(i) = i$ for $i = 1, 2$. Then, $f^{SD\sigma}(\succeq) = (0.26, 0.74)$. Since $\widehat{u}_2 = 0.52$, $f^{SD\sigma}$ is not individually rational. Moreover, since $f^{SD\sigma}(\succeq_1, \succeq'_2) = (0, 0)$, where $p'_2 = 0$, agent 2 manipulates $f^{SD\sigma}$ at profile $\succeq$ via $\succeq'_2$. Hence, $f^{SD\sigma}$ is not strategy-proof. ■

References

[1] P. Amorós. "Single-peaked preferences with several commodities," *Social Choice and Welfare* 19, 57–67 (2002).

[2] S. Barberà, M.O. Jackson and A. Neme. "Strategy–proof allotment rules," *Games and Economic Behavior* 18, 1–21, (1997).

[3] G. Bergantiños, J. Massó, and A. Neme. "The division problem with voluntary participation," *Social Choice and Welfare* 38, 371–406 (2012a).

[4] G. Bergantiños, J. Massó, and A. Neme. "The division problem with maximal capacity constraints". *SERIEs* 3, 29–57 (2012b).

[5] G. Bergantiños, J. Massó, and A. Neme. "The division problem under constraints". *Games and Economic Behavior* 89, 56–77 (2015).

[6] S. Ching. "A simple characterization of the uniform rule," *Economics Letters* 40, 57–60 (1992).

[7] S. Ching. "An alternative characterization of the uniform rule," *Social Choice and Welfare* 11, 131–136 (1994).

[8] S. Kim, G. Bergantiños and Y. Chun. "The separability principle in single-peaked economies with voluntary participation," *Mathematical Social Sciences* 78, 69–75 (2015).

[9] V. Manjunath. "When too little is as good as nothing at all: rationing a disposable good among satiable people with acceptance thresholds," *Games and Economic Behavior* 74, 576–587 (2012).

[10] T. Sönmez. "Consistency, monotonicity and the uniform rule," *Economics Letters* 46, 229–235 (1994).

[11] Y. Sprumont. "The division problem with single-peaked preferences: a characterization of the uniform allocation rule," *Econometrica* 59, 509–519 (1991).

[12] W. Thomson. "Resource-monotonic solutions to the problem of fair division when preferences are single-peaked," *Social Choice and Welfare* 11, 205–223 (1994a).

[13] W. Thomson. "Consistent solutions to the problem of fair division when preferences are single-peaked," *Journal of Economic Theory* 63, 219–245 (1994b).

[14] W. Thomson. "Population-monotonic solutions to the problem of fair division when preferences are single-peaked," *Economic Theory* 5, 229–246 (1995).

[15] W. Thomson. "The replacement principle in private good economies with single-peaked preferences," *Journal of Economic Theory* 76, 145–168 (1997).

[16] W. Thomson. "Fair allocation rules," In *Handbook of Social Choice and Welfare*, Vol. 2 (K. Arrow, A. Sen and K. Suzumura, eds.), North-Holland, Amsterdam (2010).