ECOBAS Working Papers
2016 - 07

Title:

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Abstract

Consider a group of agents located at different geographical places that are interested in some resource provided by a common supplier. Agents can be served directly from the supplier or indirectly through other agents or public switches. The first problem that we need to solve is how to provide all the agents with the resource with a minimal cost. This problem is known as the Steiner tree problem and it is NP-hard. So, in order to find a solution, heuristics need to be applied. In this paper, we allocate the cost of constructing this approximated tree among the agents in a stable way.

Keywords: Steiner tree problem, cooperative games

1 Introduction

Consider a group of agents located at different geographical places that are interested in some resource that can only be provided by a common supplier. Agents can be served directly from the supplier or indirectly through other agents or some public switches. The connection between two agents, an agent and the source, an agent and a switch, or a switch and the source entail some cost. The first problem that we need to solve is how to provide all the agents with the resource with a minimal total cost. This problem is known as the Steiner tree problem. A comprehensive overview of results on the Steiner tree problem is given in Part II of the book by Hwang et al. (1992).

This kind of problems generalize the classical minimum cost spanning tree problem where no public nodes are allowed. Kruskal (1956) and Prim (1957) designed two algorithms for obtaining a minimal cost spanning tree in polynomial time. Moreover, Bird (1976) associated with each minimum cost spanning tree problem a cooperative game with transferable utility. Granot and Huberman (1981 and 1984) studied the core of this game and proved that it is never empty.

However, the results obtained in the case of minimum cost spanning tree problems cannot be extended to the case of the Steiner tree problems. It is well-known that the Steiner tree problem is computationally intractable since it is an NP-hard problem (Garey et al, 1977). Moreover, Meggido (1976) showed that the core of the cooperative game with transferable utility related to the Steiner tree problem can be empty.

Since the Steiner tree problem is NP-hard, in many practical cases it is necessary to apply heuristic algorithms to find a good solution. Assume that using some software we obtain a tree that connects all the agents with the source that might use some public switches. Given this situation, the question that arises is: how do we allocate the cost of constructing this approximated minimum Steiner tree among the agents? In this paper we study the case of a single public node and address the problem of allocating the total cost among the agents in a stable way. Our final goal is to show what happens for any number of public nodes.

2 Classical minimum cost spanning tree problems

Let $\mathcal{N} \subset \mathbb{N} = \{1, 2, \ldots\}$ be the set of all possible agents. Given a finite subset $N \subset \mathcal{N}$, we deal with networks whose nodes are elements of a set $N_0 = N \cup \{0\}$, where N is the set of agents and 0 is a special node called the *source*. We assume $N = \{1, \ldots, n\}$.

A *cost matrix* $C = (c_{ij})_{i,j \in N_0}$ gives the cost of a direct link between any two nodes. We assume symmetric costs, *i.e.*, for each $i, j \in N_0$, $c_{ij} = c_{ji} \geq 0$ and for each $i \in N_0$, $c_{ii} = 0$.

We denote the set of all cost matrices with agent set N by $\mathcal{C}^N$. Given C, $C' \in \mathcal{C}^N$ we say that $C \leq C'$ if for each $i, j \in N_0$, $c_{ij} \leq c'_{ij}$.

A *minimum cost spanning tree problem*, briefly referred to as an *mcstp*, is a pair (N_0, C) where $N \subset \mathcal{N}$ is a finite set of agents, 0 is the source, and $C \in \mathcal{C}^N$ is a cost matrix. Given an *mcstp* (N_0, C) and $S \subset N$, we denote the restriction of the *mcstp* to $S_0 = S \cup \{0\}$ by (S_0, C).

A *network* g over N_0 is a subset of $\{(i, j) : i, j \in N_0, i \neq j\}$. The elements of g are called *arcs*. Since we assume symmetric costs, we work with undirected arcs, *i.e.*, $(i, j) = (j, i)$.

Given a network g and a pair of nodes i and j, a *path from i to j in g* is a sequence of distinct arcs $g_{ij} = \{(i_{s-1}, i_s)\}_{s=1}^{p}$ that satisfy $(i_{s-1}, i_s) \in g$ for each $s \in \{1, 2, \ldots, p\}$, $i = i_0$ and $j = i_p$. A *cycle* is a path from i to i.

A *tree* over N_0 is a network such that for each $i \in N$, there is a unique path from i to the source.

We denote the set of all networks over N_0 by $\mathcal{G}^N$ and the set of networks over N_0 in such a way that every agent in N is connected to the source by

$\mathcal{G}_0^N$.

Given an *mcstp* (N_0, C) and $g \in \mathcal{G}^N$, we define the *cost* of g as

$$c(g, C) = \sum_{(i,j)\in g} c_{ij}.$$

A *minimal tree* for (N_0, C), briefly referred to as *mt*, is a tree $t \in \mathcal{G}_0^N$ such that $c(t, C) = \min_{g\in\mathcal{G}_0^N} c(g, C)$. A *mt* always exists, although it may not be unique. Given an *mcstp* (N_0, C), $m(N_0, C)$ denotes the cost of any *mt* t in (N_0, C).

After obtaining a *mt*, one of the most important issues addressed in the literature on *mcstp* is how to divide its cost $m(N_0, C)$ among the agents. A *cost allocation rule* is a map ψ that associates with each *mcstp* (N_0, C) a vector $\psi(N_0, C) \in \mathbb{R}^N$ such that $\sum_{i\in N} \psi_i(N_0, C) = m(N_0, C)$. Given an agent i, $\psi_i(N_0, C)$ denotes its payment.

Bird (1976), proposes an allocation (Bird's allocation) associated with a minimal tree. In this allocation, each agent pays the direct cost of connection that provides him with the resource. Formally, given a *mt* $t = \{(i, \pi^t(i))\}_{i\in N}$, where $\pi^t(i)$ denotes the first agent in the path from agent i to the source 0 in the tree, Bird proposes the allocation $B_i(N_0, C) = c_{i\pi^t(i)}$ for all $i \in N$.

3 The Model

The *minimum cost Steiner tree problem*, briefly *mcStp*, is defined as a triple $(N_0^P, \hat{t}, C)$ where $N = \{1, 2, \ldots, n\}$ is the set of agents; 0 is the source; $P \subset \mathcal{N} \setminus N$ is a finite set of public nodes, $N_0^P = N \cup P \cup \{0\}$; $\hat{t} = \left\{\left(i, \pi^{\hat{t}}(i)\right)\right\}_{i\in N\cup P}$ is a Steiner tree, not necessarily the optimal one, obtained using heuristics, and C is a non-negative cost matrix where c_{ij} is the cost associated with the arc $(i, j) \in \hat{t} \cup \{(i, j) \notin \hat{t} : i, j \in N_0\}$. We assume that the tree $\hat{t}$ uses all the public nodes in P. We assume that the costs are symmetric, i.e., $c_{ij} = c_{ji}$.

The concepts of tree, path, and cost of a tree can be defined in the same way as in minimum cost spanning tree problems. However, in this case, a network involves not only the arcs in $\{(i,j) : i,j \in N_0\}$ but also the arcs in $\hat{t}$.

We assume that $\hat{t}$ is an optimal tree, i.e.

$$c\left(\hat{t}, C\right) = \min_{Q \subset P} m\left(N_0^P, C\right).$$

Remark 3.1 *Note that we only know the cost of connection between two different agents $i,j \in N_0$ and the costs of the arcs in $\hat{t}$. Therefore, C is not a complete matrix.*

This problem can be represented as a graph as the following example shows.

Example 3.1 *Consider the graph of Figure 1. This graphs represents the minimum cost Steiner tree problem $(N_0^P, \hat{t}, C)$ where $N = \{1,2,3\}$;$P = \{p\}$, $\hat{t} = \{(0,3), (3,p), (p,1), (p,2)\}$; and where the costs of matrix C are given by the numbers indicated over the arcs.*

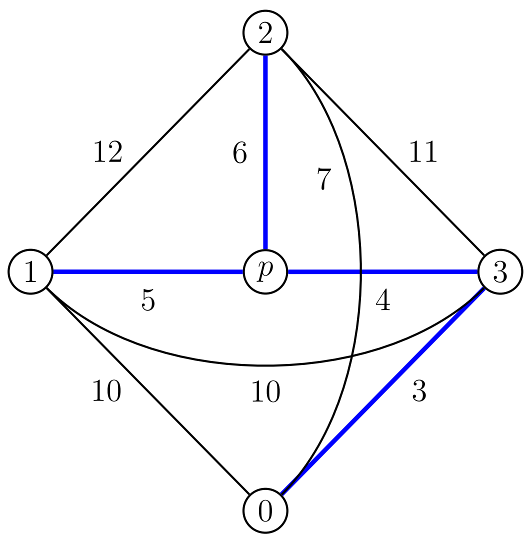

Figure 1: *mcStp* $(N_0^P, \hat{t}, c)$

Note that the main difference between this problem and the classical one is not only the presence of the tree $\hat{t}$ and the public nodes, but also that the network is not complete. There are arcs that are missing.

The tree $\hat{t}$ spans all the agents in N_0, but it also contains the switch nodes P outside N_0. These switches are public nodes. Besides $c(\hat{t}) \leq c(t)$ for each tree t that spans all the agents in N_0, i.e., $\hat{t}$ is a minimal tree (mt).

Given a tree t, we define the following sets:

- $\rho^t(i)$ denotes the set of nodes in the unique path in t from node i to the source 0, and $\pi^t(i)$ denotes the first node after i in the unique path $\rho^t(i)$ from node i to the source. Notice that $i \in \rho^t(i)$ but $i \neq \pi^t(i)$.
- $F(i,t) = \{j \in N_0^P : i \in \rho^t(j)\}$ denotes the set of followers of node i in t. In particular, $i \in F(i,t)$. Let $F^*(i,t) = \left\{j \in N^P : \pi^t(j) = i\right\}$ denote the set of immediate followers of node i. Notice that $F^*(i,t) \subset F(i,t)$.

Note that since we know the tree that is going to be constructed in order to provide all the agents with the resource, the Steiner tree $\hat{t}$, the problem that arises next is how to allocate the cost of constructing $\hat{t}$ among the agents. A *cost allocation rule* is a map ψ that associates with each *mcStp* $(N_0^P, \hat{t}, C)$ a vector $\psi(N_0^P, \hat{t}, C) \in \mathbb{R}^N$ such that $\sum_{i \in N} \psi_i(N_0^P, \hat{t}, C) = c(\hat{t}, C)$. Given an agent i, $\psi_i(N_0^P, \hat{t}, C)$ denotes its payment.

Given a *mcStp* $(N_0^P, \hat{t}, C)$, we define the Core of $\left(N_0^P, \hat{t}, C\right)$ as the set of payments that cover the cost of $\hat{t}$ and cannot be improved by any subcoalition of agents. Namely,

$$Core\left(N_0^P, \hat{t}, C\right) = \left\{ \begin{array}{c} x \in \mathbb{R}^N : \sum_{i \in N} x_i = c(\hat{t}, C) \text{ and} \\ \sum_{i \in S} x_i \leq \min_{Q \subset P} m\left(S_0^Q, C\right) \text{ for all } S \subset N \end{array} \right\}.$$

In Example 3.1, the core is non-empty. For instance, the allocation $(8,7,3) \in Core(N_0^P, \hat{t}, C)$. Does this result hold in general for the class of minimum cost Steiner tree games?

Note that in Example 3.1, if we only consider the Steiner tree $\hat{t}$, there are two natural allocations. The first one following the idea of Bird's allocation, the agents pay the arcs that provide them with the resource and split equally the cost of the arc $(p,3)$, $(5+2, 6+2, 3) = (7,8,3)$. The second one is an allocation in terms of arc use, i.e., the agents pay equally for the arcs they use in order to enjoy the resource, $(5+2+1, 6+2+1, 1) = (8, 9, 1)$. Note that in this case neither allocation belongs to the core. This is because we are omitting information by not considering the rest of the costs in the graph. Note that agent 2 will not pay more than 7 in order to enjoy the resource. Thus, these kind of problems are not equivalent to fixed tree problems.

Our main objective is to study if the core of mcStp is non empty.

Theorem 3.1 *Given a minimum cost Steiner tree problem, its core is nonempty.*

Proof. Fix $\left(N_0^P, \hat{t}, C\right)$. We will prove the following (stronger) result: there exists a core element $x \in \mathbb{R}_+^N$ satisfying

$$x_i = c_{i\pi^{\hat{t}}(i)} \text{ for all } i \in N \text{ such that } \pi^{\hat{t}}(i) \in N_0, \tag{1}$$

$$\sum_{i \in F^*(p,\hat{t})} x_i = \sum_{i \in F^*(p,\hat{t})} c_{i\pi^{\hat{t}}(i)} + c_{p\pi^{\hat{t}}(p)} \text{ for all } p \in P, \tag{2}$$

and

$$c_{i\pi^{\hat{t}}(i)} \leq x_i \leq c^*_{i\pi^{t^*_p}(i)} \text{ for all } i \in N \text{ such that } \pi^{\hat{t}}(i) = p \in P \tag{3}$$

where t_p^* is a *mt* in the *mcstp* $(F^*(p,\hat{t})_{\pi^{\hat{t}}(p)}, C^*)$ defined as

$$c^*_{kl} = \min_{\substack{i \in F(k,\hat{t}) \\ j \in F(l,\hat{t})}} c_{ij}$$

for all $k, l \in F^*(p,\hat{t})$, and

$$c^*_{k\pi^{\hat{t}}(p)} = \min_{\substack{i \in F(k,\hat{t}) \\ j \in N_0^P \setminus F(p,\hat{t})}} c_{ij}$$

for all $k \in F^*(p,\hat{t})$.

Moreover, every x satisfying (1), (2) and (3) belongs to the core.

We prove this result by induction in the number of public nodes $|P|$.

Consider $|P| = 0$. In this case the problem $(N_0^P, \hat{t}, C)$ coincides with the classical minimum cost spanning tree (N_0, C), where $\hat{t}$ is a mt of this problem. Consider $x = (c_{i\pi^{\hat{t}}(i)})_{i \in N}$. Note that x clearly satisfies (1), (2), and (3). Moreover, x is the unique allocation satisfying those conditions. We only need to prove that it is a core element. The allocation x is the Bird allocation associated with the minimal tree $\hat{t}$. Since the Bird allocation is always in the core, $x \in Core(N_0, C) = Core(N_0^P, \hat{t}, C)$.

Assume the result holds when $|P| < r$ and consider the Steiner problem $(N_0^P, \hat{t}, C)$ where $|P| = r$. Fix $p \in P$ such that $P \cap F(p, \hat{t}) = \{p\}$ (notice that we can always find such a p). Let $F^* := F^*(p, \hat{t}) \subset N$ be the set of immediate followers of p in $\hat{t}$. Let $p_0 := \pi^{\hat{t}}(p)$ be the immediate predecessor of p. We define the *mcstp* $(F^*_{p_0}, C^*)$ as before. Let t^* be a *mt* in $(F^*_{p_0}, C^*)$. We check that:

$$c_{kp} \leq c^*_{k\pi^{t^*}(k)} \text{ for all } k \in F^* \tag{4}$$

and

$$\sum_{k \in F^*} c_{kp} + c_{pp_0} \leq \sum_{k \in F^*} c^*_{k\pi^{t^*}(k)}. \tag{5}$$

We first prove (4). Assume there exists some $k \in F^*$ with $c_{kp} > c^*_{k\pi^{t^*}(k)}$. Let (i, j) be the corresponding arc with $c_{ij} = c^*_{k\pi^{t^*}(k)}$. Then $(i, j) \notin \hat{t}$ and moreover $\left(\hat{t} \setminus \{(k, p)\}\right) \cup \{(i, j)\}$ is a spanning tree in C with a cost strictly less than $\hat{t}$'s, which is impossible because $\hat{t}$ is optimal.

We now prove (5). Let g be a graph in (N_0, C) associated to t^*. Namely for each $(k, l) \in t^*$ we take $(i, j) \in g$ such that $i \in F\left(k, \hat{t}\right)$. When $l \in F^*$ we take $j \in F\left(l, \hat{t}\right)$ such that $c_{ij} = c^*_{kl}$. When $l = p_0$ we take $j \in N_0^P \setminus F(p, \hat{t})$ such that $c_{ij} = c^*_{kl}$. Let $\hat{t}^{-p} := \left\{(i, j) \in \hat{t} : i, j \neq p\right\}$. Then $g \cup \hat{t}^{-p}$ is a spanning tree that connects all the nodes in N_0 to the source. Since $\hat{t}$ is optimal, we have $c\left(\hat{t}, C\right) \leq c\left(g \cup \hat{t}^{-p}, C\right)$. By simplifying the arcs in $\hat{t}^{-p}$ (since they belong to both trees), we get

$$\sum_{k \in F^*} c_{kp} + c_{pp_0} \leq \sum_{(i,j) \in g} c_{ij}.$$

But $\{c_{ij}\}_{(i,j) \in g} = \left\{c^*_{k\pi^{t^*}(k)}\right\}_{k \in F^*}$ and hence we get (5).

Under (4) and (5), we can find $e \in \mathbb{R}^{F^*}$ satisfying the following two conditions:

$$c_{kp} \leq e_k \leq c^*_{k\pi^{t^*}(k)} \tag{6}$$

for all $k \in F^*$, and

$$\sum_{k \in F^*} c_{kp} + c_{pp_0} = \sum_{k \in F^*} e_k. \tag{7}$$

Let $t' := \hat{t}^{-p} \cup t^*$. We define the *mcStp* $\left(N_0^{P\setminus\{p\}}, C'\right)$ as $c'_{k\pi^{t^*}(k)} = e_k$ for all $k \in F^*$, and $c'_{ij} = c_{ij}$ otherwise. It is straightforward to check that $c'_{ij} \leq c_{ij}$ for all $i, j \in N_0^{P\setminus\{p\}}$. We prove that $c\left(\hat{t}, C\right) = c(t', C')$ and that t' is an optimal tree in $\left(N_0^{P\setminus\{p\}}, C'\right)$. We first check that $c\left(\hat{t}, C\right) = c(t', C')$:

$$\begin{aligned} c(t', C') &= c\left(\hat{t}^{-p}, C'\right) + c\left(t^*, C'\right) = c\left(\hat{t}^{-p}, C\right) + c\left(t^*, C'\right) & (8) \\ &= c\left(\hat{t}, C\right) - \sum_{k \in F^*} c_{kp} - c_{pp_0} + \sum_{k \in F^*} e_k \overset{(7)}{=} c\left(\hat{t}, C\right). & (9) \end{aligned}$$

We know prove that t' is an optimal tree in $\left(N_0^{P\setminus\{p\}}, C'\right)$. Assume it is not. Then, there exists a spanning tree t in $N_0^{P\setminus\{p\}}$ such that $c(t, C') < c(t', C')$. Let $\tilde{t}$ be such a tree with a maximum number of arcs in t^*, i.e. $|t \cap t^*| > \left|\tilde{t} \cap t^*\right|$ implies either t is not a spanning tree in $N_0^{P\setminus\{p\}}$ or $c\left(t, C'\right) \geq c(t', C')$.

If $t^* \subset t$, then

$$t'' := (t \setminus t^*) \cup \{(i, p)\}_{i \in F^*} \cup \{(p, p_0)\}$$

is a spanning tree in N_0^P with the same cost as t (because of (7) and that $c_{i\pi^{\hat{t}}(i)} = c'_{i\pi^{\hat{t}}(i)}$ for all $i \in F^*$). Thus,

$$c(t'', C) = c(t, C') < c(t', C') = c(\hat{t}, C),$$

which is a contradiction because $\hat{t}$ is an optimal tree.

Hence, there exists $(k^*, l^*) \in t^*$ with $l^* = \pi^{t^*}(k^*)$ such that $(k^*, l^*) \notin t$.

Since $(k^*, l^*) \notin t$, we have that $t \cup \{(k^*, l^*)\}$ has a cycle g^c in $N_0^{P\setminus\{p\}}$ containing arc (k^*, l^*). Let $(i^c, j^c) \in g^c$ be the first arc in the (unique) path in t between k^* and l^* satisfying $i^c \in F\left(i^*, \hat{t}\right)$ for some $i^* \in F\left(k^*, t^*\right)$ and

$j^c \notin F\left(i, \hat{t}\right)$ for all $i \in F(k^*, t^*)$. Under our asumptions, $(i^c, j^c) \notin t^*$ because otherwise $(i^c, j^c) = (k^*, l^*)$, whith is impossible.

Since $(i^c, j^c) \notin t^*$ and t has a maximum number of arcs in t^*, $c_{i^c j^c} = c'_{i^c j^c} < c'_{k^* l^*}$.

Let $j^* \in F^*_{p_0} \setminus \{k^*\}$ such that $j^c \in F\left(j^*, \hat{t}\right)$. If $j^c \in N_0^{P\setminus\{p\}} \setminus F(p, \hat{t})$, we take $j^* = p_0$. Let

$$t^1 := (t^* \setminus \{(k^*, l^*)\}) \cup \{(i^*, j^*)\}.$$

Clearly, t^1 is a spanning tree in $\left(F^*_{p_0}, C^*\right)$. Moreover,

$$c(t^1, C^*) = c(t^*, C^*) - c^*_{k^* l^*} + c^*_{i^* j^*}.$$

Since t^* is a *mt* in $\left(F^*_{p_0}, C^*\right)$, we deduce $-c^*_{k^* l^*} + c^*_{i^* j^*} \geq 0$, i.e. $c^*_{i^* j^*} \geq c^*_{k^* l^*}$. On the other hand, by definition of C^*, $c_{i^c j^c} \geq c^*_{i^* j^*}$ and hence $c_{i^c j^c} \geq c^*_{k^* l^*}$. By definition of C', $c^*_{k^* l^*} \geq c'_{k^* l^*}$, and hence $c_{i^c j^c} \geq c'_{k^* l^*}$, which is a contradiction.

We have then that t' is an optimal tree in $\left(N_0^{P\setminus\{p\}}, C'\right)$. By induction hypothesis, there exists some $x \in Core\left(N_0^{P\setminus\{p\}}, t', C'\right)$ satisfying (1), (2) and (3) in C'. Besides, any allocation satisfying those conditions belongs to the core.

We prove that x satisfies (1), (2) and (3) in C.

We prove that x satisfies (1) in C. Remark that $c_{i\pi^{\hat{t}}(i)} = c'_{i\pi^{\hat{t}}(i)}$ and $\pi^{\hat{t}}(i) = \pi^{t'}(i)$ for all $i \in N \setminus F^*$. Hence, given $i \in N$ with $\pi^{\hat{t}}(i) \in N_0$,

$$x_i = c'_{i\pi^{t'}(i)} = c'_{i\pi^{\hat{t}}(i)} = c_{i\pi^{\hat{t}}(i)}.$$

We know prove that x satisfies (2) in C. Given $q \in P$, we distinguish two cases:

- If $q = p$, $\sum\limits_{i \in F^*(p,\hat{t})} x_i = \sum\limits_{i \in F^*(p,\hat{t})} e_i \overset{(7)}{=} \sum\limits_{i \in F^*(p,\hat{t})} c_{i\pi^{\hat{t}}(i)} + c_{pp_0}$.
- If $q \neq p$, it is clear that $F^*(q, \hat{t}) = F^*(q, t')$ and $c'_{i\pi^{t'} i} = c_{i\pi^{\hat{t}}(i)}$ for all $i \in F^*(q, t')$. Thus,

$$\sum_{i \in F^*(q,\hat{t})} x_i = \sum_{i \in F^*(q,t')} x_i = \sum_{i \in F^*(q,t')} c'_{i\pi^{t'}(i)} + c'_{qq_0} = \sum_{i \in F^*(q,\hat{t})} c_{i\pi^{\hat{t}}(i)} + c_{qq_0}$$

We now prove that x satisfies (3) in C. Given $i \in N$ with $\pi^{\hat{t}}(i) \in P$, we distinguish two cases:

- If $\pi^{\hat{t}}(i) = p$, then $x_i = e_i$ and (3) follows from (6).
- If $\pi^{\hat{t}}(i) = q \neq p$, then $c_{i\pi^{\hat{t}}(i)} = c'_{i\pi^{t'}(i)} \leq x_i \leq c'^{*}_{i\pi^{t'^{*}_{q}}(i)} = c^{*}_{i\pi^{t^{*}_{q}}(i)}$.

Finally, we prove that any allocation y that satisfies (1), (2) and (3) in C, belongs to the core: $y \in Core\left(N_0^P, \hat{t}, C\right)$. In particular, $x \in Core(N_0^P, \hat{t}, C)$.

Consider the mcStp $(N_0^{P\setminus\{p\}}, t', C^y)$, where $c^y_{i\pi^{t'}(i)} = y_i$ if $i \in F^*$ and $c^y_{ij} = c_{ij}$ otherwise. Since y salsifies (2) and (3), $\sum\limits_{i\in F^*} y_i = \sum\limits_{i\in F^*} c_{i\pi^{\hat{t}}(i)} + c_{pp_0}$ and $c_{i\pi^{\hat{t}}(i)} \leq y_i \leq c^{*}_{i\pi^{t^*}(i)}$. Thus, it can be proved that the mcStp $(N_0^{P\setminus\{p\}}, t', C^y)$ is well defined. Besides, it can be easily proved that y satisfies conditions (1), (2) and (3) in C^y. Thus, by the induction hypothesis, it is a core element of $(N_0^{P\setminus\{p\}}, t', C^y)$. Next we prove $y \in Core(N_0^P, \hat{t}, C)$.

Since $y \in Core\left(N_0^{P\setminus\{p\}}, t', C^y\right)$,

$$\begin{aligned}\sum_{i\in N} y_i &= m(N_0^{P\setminus\{p\}}, C^y) = c(t', C^y) \\ &= c(\hat{t}^{-p}, C) + c(t^*, C^y) = c(\hat{t}^{-p}, C) + \sum_{i\in F^*} y_i \\ &\overset{(7)}{=} c(\hat{t}^{-p}, C) + \sum_{i\in F^*} c_{i\pi^{\hat{t}}(i)} + c_{pp_0} = c(\hat{t}, C).\end{aligned}$$

Let $S \subset N$. We have to prove that $\sum\limits_{i\in S} y_i \leq \min\limits_{Q\subset P} m(S_0^Q, C)$. Since $y \in Core(N_0^{P\setminus\{p\}}, C^y)$, we have $\sum\limits_{i\in S} y_i \leq \min\limits_{Q\subset P\setminus\{p\}} m(S_0^Q, C^y)$. But $C^y \leq C$, and hence $m(S_0^Q, C^y) \leq m(S_0^Q, C)$ for all $Q \subset P \setminus \{p\}$. As a result, $\sum\limits_{i\in S} y_i \leq \min\limits_{Q\subset P\setminus\{q\}} m(S_0^Q, C)$.

We still have to prove that $\sum\limits_{i\in S} y_i \leq m(S_0^Q, C)$ when $p \in Q$. Fix $Q \subset P$, such that $p \in Q$. Let t^S an optimal tree in (S_0^Q, C). We will prove that $\sum\limits_{i\in S} y_i \leq c(t^S, C)$.

If t^S does not use the public node p, then t^S is a spanning tree in $(S_0^{Q\setminus\{p\}}, C)$ and hence $c(t^S, C) = m(S_0^{Q\setminus\{p\}}, C) \geq m(S_0^{P\setminus\{p\}}, C) \geq \sum_{i \in S} y_i$.

Assume now t^S uses the public node p. We define the $mcStp$ $((N \cup \{p\})_0^{P\setminus\{p\}}, C^p)$ obtained from C by assuming that the connection costs of each node $i \notin F^* \cup \{p_0\}$ to p is very high and that the rest of the costs in C^p are given by C. Formally, $c_{pi}^p = 2^n \max\{c_{kl} : \{k, l\} \subset N_0^P\}$ when $i \notin F^* \cup \{p_0\}$ and $c_{ij}^p = c_{ij}$ otherwise.

It is obvious that t^S is a tree in the subproblem $((S \cup \{p\})_0^{Q\setminus\{p\}}, C^p)$. Thus,

$$m((S \cup \{p\})_0^{Q\setminus\{p\}}, C^p) \leq c(t^S, C^p) = c(t^S, C) = m(S_0^Q, C). \qquad (10)$$

Next we prove that $\hat{t}$ is a mt in $((N \cup \{p\})_0^{P\setminus\{p\}}, C^p)$. Suppose not. Then, there exists a tree t in $((N \cup \{p\})_0^{P\setminus\{p\}}, C^p)$ such that $c(t, C^p) < c(\hat{t}, C^p)$. Because of the definition of C^p given (p, i) with $i \notin F^* \cup \{p_0\}$ we have that $(p, i) \notin t$. Thus, t is also a tree in $(N_0^P, \hat{t}, C)$ and $c(t, C) = c(t, C^p)$. Since $c(\hat{t}, C) = c(\hat{t}, C^p)$, we have that $c(t, C) < c(\hat{t}, C)$, which contradicts that $\hat{t}$ is a mt in $(N_0^p, \hat{t}, C)$.

By the induction hypothesis, we know there exists an element in $Core((N \cup \{p\})_0^{P\setminus\{p\}}, C^p)$ and that every allocation satisfying conditions (1), (2) and (3) in C^p belongs to the core. We define an allocation $y' \in \mathbb{R}^{N\cup\{p\}}$ as

$$y'_i = \begin{cases} c^p_{i\pi^{\hat{t}}(i)} = c_{i\pi^{\hat{t}}(i)} & \text{if } i \in F^* \cup \{p\} \\ y_i & \text{otherwise} \end{cases}$$

Next we prove that the allocation y' satisfies conditions (1), (2) and (3) in C^p.

We need to prove that $y'_i = c_{i\pi^{\hat{t}}(i)}$ for all $i \in N \cup \{p\}$ such that $\pi^{\hat{t}}(i) \in N_0 \cup \{p\}$. Note that (1) in C^p is satisfied for all $i \in N$ with $\pi^{\hat{t}}(i) \in N_0$ because y satisfies (1) in C^y. Condition (1) in C^p is satisfied for $i \in F^* \cup \{p\}$ by definition of y'.

As for (2), we need to prove, for each $q \in P \setminus \{p\}$,

$$\sum_{j \in F^*(q,\hat{t})} y'_j = \sum_{j \in F^*(q,\hat{t})} c^p_{j\pi^{\hat{t}}(j)} = \sum_{j \in F^*(q,\hat{t})} c_{j\pi^{\hat{t}}(j)}.$$

This equality is true because $y'_i = y_i$ for all $i \in F^*(q,\hat{t})$ and y satisfies (2) in C^y.

As for (3), given $i \in N \cup \{p\}$ such that $\pi^{\hat{t}}(i) \in P \setminus \{p\}$, we have to prove

$$c_{i\pi^{\hat{t}}(i)} \leq y'_i \leq c^*_{i\pi^{t^*}(i)}.$$

Since there are no consecutive public nodes, it follows from the fact that (3) is satisfied by y in C^y.

The allocation y' satisfies conditions (1), (2) and (3) in C^p. Thus, it is a core element of $((N \cup \{p\})_0^{P\setminus\{p\}}, C^p)$,

$$\sum_{i\in S\cup\{p\}} y'_i \leq m((S \cup \{p\})_0^{Q\setminus\{p\}}, C^p).$$

By definition of C^y it is straightforward to check that $y_i = c^y_{i\pi^{t'}(i)} \geq c_{i\pi^{\hat{t}}(i)}$ when $i \in F^*$, and $\sum_{i\in F^*}(c^y_{i\pi^{t'}(i)} - c_{i\pi^{\hat{t}}(i)}) = c_{pp_0}$. So,

$$\begin{aligned}
\sum_{i\in S} y_i &= \sum_{i\in S\setminus F^*} y'_i + \sum_{i\in S\cap F^*} y_i \\
&= \sum_{i\in S\setminus F^*} y'_i + \sum_{i\in S\cap F^*} y'_i + \sum_{i\in S\cap F^*}(y_i - y'_i) \\
&= \sum_{i\in S} y'_i + \sum_{i\in S\cap F^*}(c^y_{i\pi^{t'}(i)} - c_{i\pi^{\hat{t}}(i)}) \\
&\leq \sum_{i\in S} y'_i + \sum_{i\in F^*}(c^y_{i\pi^{t'}(i)} - c_{i\pi^{\hat{t}}(i)}) \\
&= \sum_{i\in S} y'_i + c_{pp_0} = \sum_{i\in S\cup\{p\}} y'_i \\
&\leq m((S \cup \{p\})_0^{Q\setminus\{p\}}, C^p) \overset{(10)}{\leq} m(S_0^Q, C).
\end{aligned}$$

■

4 Acknowledgments

Financial support from Ministerio de Ciencia e Innovación through grant ECO2011-23460 and from Xunta de Galicia through grants INCITE09-207-064 and 10PXIB362299PR is gratefully acknowledged.

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